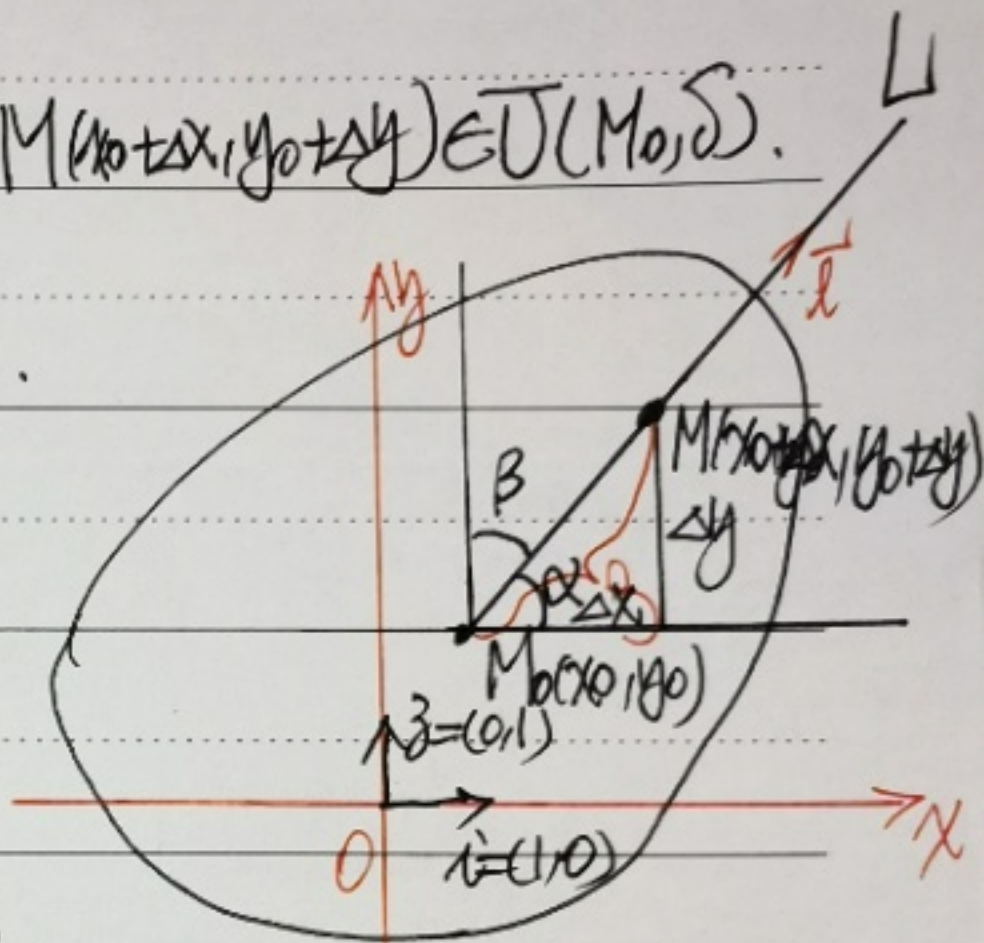


第10+1讲: 方向导数与梯度 (gradient)

(一) 函数 $u = f(x, y)$ 在点 $M_0(x_0, y_0)$ 处沿方向 \vec{l} 的方向导数 $= \frac{\partial u}{\partial l} \Big|_{M_0}$

设 $u = f(x, y)$ 在 $\alpha \in U(M_0, \delta)$ 中, $M(x_0 + \Delta x, y_0 + \Delta y) \in U(M_0, \delta)$.

$\Rightarrow M_0$ 为起点作射线 L 过 M 点.



记 $\rho = \rho(M_0, M) = |M_0M| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$,

记 $\lim_{\rho \rightarrow 0^+} \frac{f(M) - f(M_0)}{|M_0M|} = \frac{\partial u}{\partial l} \Big|_{M_0} = \frac{\partial f}{\partial l} \Big|_{M_0}$.

将 $\frac{\partial u}{\partial l} \Big|_{M_0} = \lim_{\rho \rightarrow 0^+} \frac{f(x_0 + \rho \cos \alpha, y_0 + \rho \sin \alpha) - f(x_0, y_0)}{\rho}$

$\rho = |M_0M| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

$\Delta x = \rho \cos \alpha, \Delta y = \rho \sin \alpha$

$\vec{l}^0 = (\cos \alpha, \sin \alpha), \alpha \in \left[\frac{\pi}{2}, \frac{3\pi}{2} \right]$

$= (\cos \alpha, \sin \alpha)$.

是函数 $u = f(x, y)$ 在点 M_0 处沿方向 \vec{l} 的

方向导数, 表示 u 关于 \vec{l} 方向在 M_0 处的变化率.

例1. 若 $f'_x(M_0) = f'_x(x_0, y_0)$ 存在, 则 $u = f(x, y)$ 在 M_0 处沿

x轴正向: $\vec{l} = (1, 0)$ 的方向导数为 $\frac{\partial u}{\partial x} \Big|_{M_0} = \lim_{\Delta x \rightarrow 0^+} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$

$= \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} = f'_x(x_0, y_0)$; 而 $u = f(x, y)$ 在 M_0 处沿

方向导数: directional derivative

(1)

x轴负向: $\vec{l} = (-1, 0) = -i$ 的方向导数 $\frac{\partial u}{\partial l} \Big|_{M_0} =$

$$\lim_{\Delta x \rightarrow 0^+} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{-\Delta x} = -\lim_{\Delta x \rightarrow 0^+} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} = -f'_x(M_0)$$

同理, 当 $f'_y(x_0, y_0) = B$ (常数) 时, $u = f(x, y)$ 在 $M_0(x_0, y_0)$ 处沿 y 轴

正负两个方向导数都存在, 且 $\frac{\partial u}{\partial j} \Big|_{M_0} = f'_y(M_0)$, $\frac{\partial u}{\partial (-j)} \Big|_{M_0} = -f'_y(M_0)$.

这里 $j = (0, 1)$, $-j = (0, -1)$ 分别为 y 轴的正、负向。

例 2. $u = f(x, y) = \sqrt{x^2 + y^2}$ 在 $O(0, 0)$ 处 C^1 且 $f'_x(0, 0), f'_y(0, 0)$ 都不存在, 从而不可微. 但 $u = \sqrt{x^2 + y^2}$ 在 $O(0, 0)$ 处沿任何方向

$\vec{l}^0 = (\cos \alpha, \sin \alpha)$ 的方向导数 $\frac{\partial u}{\partial \vec{l}^0} \Big|_{O(0,0)} = \lim_{\rho \rightarrow 0^+} \frac{f(O + \rho \vec{l}^0) - f(O)}{\rho} = \lim_{\rho \rightarrow 0^+} \frac{f(\rho \cos \alpha, \rho \sin \alpha) - f(0, 0)}{\rho}$

$$= \lim_{\rho \rightarrow 0^+} \frac{(\rho \cos \alpha)^2 + (\rho \sin \alpha)^2 - 0}{\rho} = \lim_{\rho \rightarrow 0^+} \frac{\rho}{\rho} = 1.$$

例 3. 设 $u = f(x, y, z)$ 是 $\alpha \in U(M_0, \vec{l})$ 中, $M_0(x_0, y_0, z_0), M(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) \in U(M_0, \vec{l})$, $\rho = \rho(M_0, M) = |M_0 M| = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} > 0$, $\begin{cases} \Delta x = \rho \cos \alpha \\ \Delta y = \rho \sin \alpha \\ \Delta z = \rho \cos \beta \end{cases}$

$\vec{l}^0 = (\cos \alpha, \sin \alpha, \cos \beta)$ 已知, 则是 α .

$$\frac{\partial u}{\partial \vec{l}^0} \Big|_{M_0} = \lim_{\rho \rightarrow 0^+} \frac{f(M) - f(M_0)}{\rho} = \lim_{\rho \rightarrow 0^+} \frac{f(x_0 + \rho \cos \alpha, y_0 + \rho \sin \alpha, z_0 + \rho \cos \beta) - f(x_0, y_0, z_0)}{\rho}$$

• 当偏导数 $f'_x(x_0, y_0, z_0) = A$ (非零时), $u = f(x, y, z) \in M_0$ 处

沿曲线正向 $\vec{l} = (1, 0, 0)$, 反向 $-\vec{l} = (-1, 0, 0)$ 的方向导数都

存在, 且 $\frac{\partial u}{\partial x} \Big|_{M_0} = A, \frac{\partial u}{\partial (-x)} \Big|_{M_0} = -A$. 余类推.

例4. 当 $u = f(x, y)$ 在点 $M_0(x_0, y_0)$ 处可微时, $u \in M_0$ 处沿

• 任一方向 $\vec{l} = (\cos \alpha, \sin \alpha)$ 的方向导数 $\frac{\partial u}{\partial \rho} \Big|_{M_0}$ 都存在, 且

$$\frac{\partial u}{\partial \rho} \Big|_{M_0} = f'_x(M_0) \cos \alpha + f'_y(M_0) \sin \alpha = f'_x(M_0) \cos \alpha + f'_y(M_0) \sin \beta, \quad (\alpha + \beta = \frac{\pi}{2})$$

(★)

证: 设 $M_0(x_0 + \Delta x, y_0 + \Delta y) \in U(M_0, \delta)$, 且

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = f'_x(M_0) \Delta x + f'_y(M_0) \Delta y + o(\rho), \quad \rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\Rightarrow \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)}{\rho} = f'_x(M_0) \frac{\Delta x}{\rho} + f'_y(M_0) \frac{\Delta y}{\rho} + \frac{o(\rho)}{\rho}$$

$$= f'_x(M_0) \cos \alpha + f'_y(M_0) \sin \alpha + \frac{o(\rho)}{\rho}$$

$$\text{令 } \rho \rightarrow 0^+ \text{ 得: } \frac{\partial u}{\partial \rho} \Big|_{M_0} = \lim_{\rho \rightarrow 0^+} \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)}{\rho} = f'_x(M_0) \cos \alpha + f'_y(M_0) \sin \alpha$$

同理. 当 $u = f(x, y, z)$ 在点 $M_0(x_0, y_0, z_0)$ 处可微时, $u \in M_0$ 处

• 沿任意方向 $\vec{l} = (\cos \alpha, \cos \beta, \cos \gamma)$ 的方向导数 $\frac{\partial u}{\partial \rho} \Big|_{M_0}$ 都存在

(3)

$$\bullet \frac{\partial H}{\partial \vec{r}^0} \Big|_{M_0} = f'_x(M_0) \omega \alpha + f'_y(M_0) \omega \beta + f'_z(M_0) \omega \gamma \quad (A_1)$$

在(A)中称向量 $(f'_x(M_0), f'_y(M_0))$ 为函数 $u = f(x, y)$ 在 $M_0(x_0, y_0)$

处的梯度; 在(A)中, 称向量 $(f'_x(M_0), f'_y(M_0), f'_z(M_0))$ 为函数

$u = f(x, y, z)$ 在 M_0 处的梯度, 记作:

$$\bullet \operatorname{grad} f(x_0, y_0) = (f'_x(M_0), f'_y(M_0)); \operatorname{grad} f(x_0, y_0, z_0) = (f'_x(M_0), f'_y(M_0), f'_z(M_0)).$$

$$\text{也可记作: } \operatorname{grad} f(x_0, y_0) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \Big|_{M_0}, \operatorname{grad} f(x_0, y_0, z_0) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \Big|_{M_0}$$

$u = f(x, y, z) \in U(M_0, \delta)$ 中 $s \in M$ 的梯度则记作

$$\operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) u = \nabla u.$$

其中 $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$ 称为梯度算子, 也称为 Hamilton 算子.

由此(A)可改写为:

$$\frac{\partial H}{\partial \vec{r}^0} \Big|_{M_0} = (f'_x(M_0), f'_y(M_0), f'_z(M_0)) \cdot (\omega \alpha, \omega \beta, \omega \gamma) = \nabla u \Big|_{M_0} \cdot \vec{r}^0 \quad (A_2)$$

$$= \operatorname{grad} f(x_0, y_0, z_0) \cdot \vec{r}^0 = |\operatorname{grad} f(M_0)| \cdot |\vec{r}^0| \cos(\widehat{\operatorname{grad} f(M_0), \vec{r}^0}) \quad (A_3)$$

$$\leq |\operatorname{grad} f(M_0)| = |\nabla u(M_0)|$$

等号当且仅当 \vec{r}^0 与 $\operatorname{grad} f(M_0)$ 同时取到。

(A).

• (二) 函数的梯度(陡度、倾斜度):

设 $u = f(x, y, z)$ 在点 $M_0(x_0, y_0, z_0)$ 处, 则 $f(x, y, z)$ 在 M_0 处

的梯度 $\text{grad} f(x_0, y_0, z_0) = (f'_x(M_0), f'_y(M_0), f'_z(M_0))$ 是一个向量.

这个向量的模 $|\text{grad} f(x_0, y_0, z_0)|$ 是 $f(x, y, z)$ 在点 M_0 处所有

方向的方向导数中的最大值, 而梯度的方向即是 $f(x, y, z)$

在点 M_0 处的方向导数取最大值的方向.

$$\text{故 } \frac{\partial u}{\partial l} \Big|_{M_0} = \text{grad} f(M_0) \cdot \hat{l} = |\text{grad} f(M_0)| \cos(\text{grad} f(M_0), \hat{l}) \leq |\text{grad} f(M_0)|$$

可知, 当 \hat{l} 与 $\text{grad} f(M_0)$ 一致时, $\frac{\partial u}{\partial l} \Big|_{M_0}$ 取最大值 $|\text{grad} f(M_0)|$

而当 \hat{l} 与 $\text{grad} f(M_0)$ 相反时, $\frac{\partial u}{\partial l} \Big|_{M_0}$ 取最小值 $-|\text{grad} f(M_0)|$

$$\text{即 } \left(\frac{\partial u}{\partial l} \Big|_{M_0} \right)_{\max} = |\text{grad} f(M_0)|, \left(\frac{\partial u}{\partial l} \Big|_{M_0} \right)_{\min} = -|\text{grad} f(M_0)|. \quad (\star A)$$

换言之, 在点 M_0 处, 沿梯度 $\text{grad} f(M_0)$ 的方向, $f(x, y, z)$ 的

变化率是最大的. 而沿 $-\text{grad} f(M_0)$ 的方向, $f(x, y, z)$ 的

变化率最小: $-|\text{grad} f(M_0)| \leq \frac{\partial u}{\partial l} \Big|_{M_0} \leq |\text{grad} f(M_0)|. \quad (\star B)$

● 因 $\text{grad} f(M_0) = \nabla u(M_0) \triangleq \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right) \Big|_{M_0}$, 从而有

$$- \left| \nabla u(M_0) \right| \leq \frac{\partial u}{\partial l} \Big|_{M_0} \leq \left| \nabla u(M_0) \right|$$

标量场或数量场的梯度是一种特殊的微分运算,

设 $u_1 = f_1(x, y, z)$, $u_2 = f_2(x, y, z)$ 均可微, 或 $f_1, f_2 \in C^1$, 则必有

● (1). $\nabla(Cu_1 + Gu_2) = C\nabla u_1 + G\nabla u_2$, C, G 为任意函数;

(2). $\nabla(u_1 u_2) = u_2 \nabla u_1 + u_1 \nabla u_2$;

(3). $\nabla f(u) = f'(u) \nabla u$, $f \in C^1$.

$\vec{r}(u)$: $\because C, G$ 为函数, 而 $\nabla(Cu_1 + Gu_2) \triangleq (Cu_1 + Gu_2)'_x, (Cu_1 + Gu_2)'_y,$

● $(Cu_1 + Gu_2)'_z = (Cu_1)'_x + (Gu_2)'_x, (Cu_1)'_y + (Gu_2)'_y, (Cu_1)'_z + (Gu_2)'_z$
 $= C(u_1)'_x + G(u_2)'_x, C(u_1)'_y + G(u_2)'_y, C(u_1)'_z + G(u_2)'_z = C\nabla u_1 + G\nabla u_2$

$\vec{r}(f)$: $\because f \in C^1, u \in C^1$ 或 u 可微, 而 $\nabla f(u) \triangleq (f(u))'_x, (f(u))'_y, (f(u))'_z$

$$= (f'(u)u'_x, f'(u)u'_y, f'(u)u'_z) = f'(u)(u'_x, u'_y, u'_z) = f'(u)\nabla u.$$

由此 (1), (2), (3) 可知, 哈密顿算子 ∇ 与微分算子 d 相类似。

例5. 求下列各题:

(1). ~~求~~ $z = x^2 + y^2$ 在点 $M_0(1, 2)$ 处, 沿从 $(1, 2)$ 到 $(2, 2 + \sqrt{3})$ 方向的方向导数, 并求 $\frac{\partial z}{\partial r}|_{M_0(1, 2)}$ 的最大值与最小值.

(2). 求 $z = 1 - (\frac{x^2}{a^2} + \frac{y^2}{b^2})$ 在点 $M_0(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}})$ 处, 沿曲线 $L: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

在点的内法线方向的方向导数.

(3). 求数量场 $\frac{m}{r}$ 所产生的梯度场 $\nabla \frac{m}{r}$, 其中 $m > 0$ 为常数.

$r = \sqrt{x^2 + y^2 + z^2}$ 是向量 (x, y, z) 的模.

解(1)/(10). \vec{l} 方向为 $(2-1, 2+\sqrt{3}-2) = (1, \sqrt{3}) \triangleq \vec{l} \Rightarrow \vec{l}^0 = (\frac{1}{2}, \frac{\sqrt{3}}{2}) =$

$(\cos \alpha, \cos \beta)$. 而 $\nabla z(M_0) = (z'_x(M_0), z'_y(M_0)) = (2x, 2y)|_{M_0(1, 2)} = (2, 4)$.

故 $\frac{\partial z}{\partial r}|_{M_0} = \frac{\partial z}{\partial x}|_{M_0} \cdot \cos \alpha + \frac{\partial z}{\partial y}|_{M_0} \cdot \cos \beta = 2x \cdot \frac{1}{2} + 4x \cdot \frac{\sqrt{3}}{2} = 1 + 2\sqrt{3}$.

1/(20). $\because |\nabla z(M_0)| = |(2, 4)| = \sqrt{2^2 + 4^2} = 2\sqrt{5}$. $\therefore \begin{cases} \frac{\partial z}{\partial r}|_{M_0} \max = 2\sqrt{5} \\ \frac{\partial z}{\partial r}|_{M_0} \min = -2\sqrt{5} \end{cases}$
 $-2\sqrt{5} = -|\nabla z(M_0)| \leq \frac{\partial z}{\partial r}|_{M_0} \leq |\nabla z(M_0)| = 2\sqrt{5}$. 即

解(2)/(10). L 的参数方程为 $\begin{cases} x = a \cos t \\ y = b \sin t \end{cases}, t \in [0, 2\pi]$.

(17).

• 曲线的参数为 $r(t) = (x(t), y(t)) = (a \cos t, b \sin t)$, $t \in [0, 2\pi]$, $t_0 = \frac{\pi}{4}$,

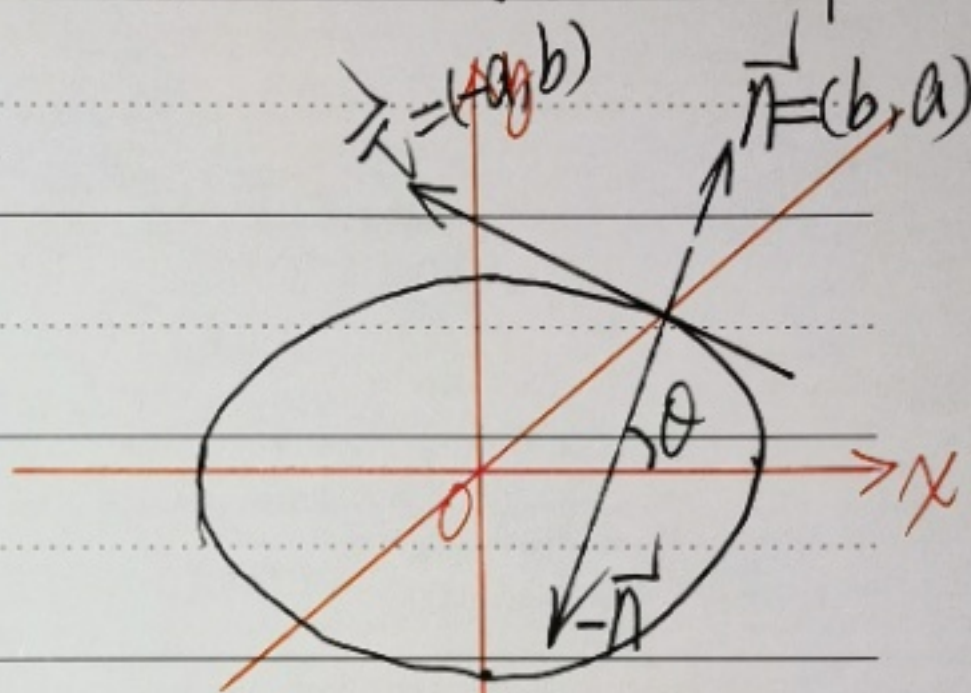
$$r'(t) = (x'(t), y'(t)) \Big|_{M_0} = (-a \sin t, b \cos t) \Big|_{M_0(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}})} = (-a \sin t, b \cos t) \Big|_{t_0 = \frac{\pi}{4}}$$

$$= (-\frac{\sqrt{2}}{2}a, \frac{\sqrt{2}}{2}b). \text{ 可取切向量 } \vec{T} = (-a, b). \text{ 则法向量 } M_0 \text{ 外}$$

法向量为 $\vec{n} = (b, a)$, 则法向量 M_0 内

• 内法向量 $\vec{l} = -\vec{n} = (-b, -a) \Rightarrow$

$$\vec{l}^0 = \frac{1}{\sqrt{a^2+b^2}} (-b, -a) = (\cos \alpha, \cos \beta)$$



$$t_0) z'_x(M_0) = -\frac{2x}{a^2} \Big|_{M_0} = -\frac{2}{a^2} \frac{a}{\sqrt{2}} = -\frac{\sqrt{2}}{a}, \quad z'_y(M_0) = -\frac{2y}{b^2} \Big|_{M_0} = -\frac{2}{b^2} \frac{b}{\sqrt{2}} = -\frac{\sqrt{2}}{b}.$$

$$\text{故 } \frac{\partial z}{\partial x} \Big|_{M_0} = z'_x(M_0) \cos \alpha + z'_y(M_0) \cos \beta = \left(-\frac{\sqrt{2}}{a}\right) \left(\frac{-b}{\sqrt{a^2+b^2}}\right) + \left(-\frac{\sqrt{2}}{b}\right) \left(-\frac{a}{\sqrt{a^2+b^2}}\right)$$

$$= \frac{\sqrt{2}}{\sqrt{a^2+b^2}} \left(\frac{b}{a} + \frac{a}{b}\right) = \frac{\sqrt{2(a^2+b^2)}}{ab}.$$

• 例(3): $\nabla \left(\frac{m}{r}\right) = \left(\frac{m}{r}\right)'_x, \left(\frac{m}{r}\right)'_y, \left(\frac{m}{r}\right)'_z$

$$\Rightarrow \left(\frac{m}{r}\right)'_x = m \left(\frac{1}{r}\right)'_x = -\frac{m}{r^2} r'_x = -\frac{m}{r^2} \frac{x}{r} = -\frac{mx}{r^3}, \text{ 同理可得}$$

$$\left(\frac{m}{r}\right)'_y = -\frac{my}{r^3}, \quad \left(\frac{m}{r}\right)'_z = -\frac{mz}{r^3}. \text{ 故 } \nabla \left(\frac{m}{r}\right) = -\frac{m}{r^3} (x, y, z)$$

$$\text{令 } \vec{r}^0 = \frac{(x, y, z)}{r}, \text{ 则 } \nabla \left(\frac{m}{r}\right) = -\frac{m \vec{r}^0}{r^2} = -\frac{m \cdot 1}{r^2} \cdot \vec{r}^0 \quad (*)$$

(8).

- (a) 右端的力学解释为: 位于原点 $O(0,0,0)$ 质量为 m 的质点, 对位于 $M(x,y,z)$ 且质量为 1 的单个质点的引力. 该引力的大小与两质点的质量乘积成正比, 而与它们之间的距离平方成反比. 并且这引力的方向由点 M 指向原点.

在物理学中, 称 $\vec{v}(\frac{m}{r}) = \frac{m \cdot 1}{r^2} (-\vec{r}_0)$ 为引力场.

这是力的向量场. 而称 $\frac{m}{r} = \frac{m}{\sqrt{x^2+y^2+z^2}}$ 为对应的

引力势函数, 简称势函数.

- 因为引力场: $\frac{m \cdot 1}{r^2} (-\vec{r}_0) = \frac{m \cdot 1}{x^2+y^2+z^2} \cdot \frac{-(x, y, z)}{\sqrt{x^2+y^2+z^2}}$

是通过势函数 $\frac{m}{r}$ 取梯度得到的. 因此, 也称这

引力场为梯度场.

(三). 作业: ex 9.2/21; 22; 23; 24; 30(2), (5); 38.

第12讲: 多元函数微分学的几何应用