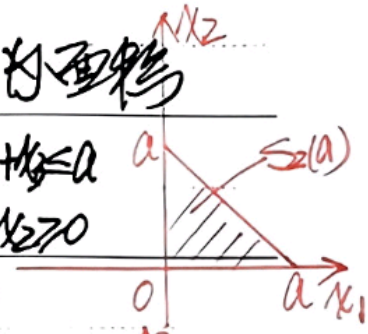
第24讲：n重积分：$\int\int\cdots\int_{\Omega_n} f(x_1,x_2,\cdots,x_n)dx_1dx_2\cdots dx_n$

(一) 求n维单形(simple shape) $S_n(a)$：

$0\le x_1+x_2+\cdots+x_n\le a$，$x_1\ge 0, x_2\ge 0,\cdots,x_n\ge 0$，$(a>0)$ 的"体积"或测度(measure) $V(S_n(a))$

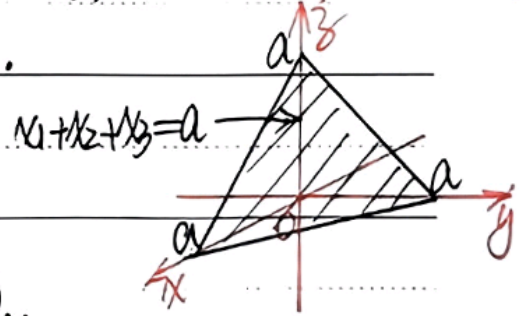
(1°) 一维单形 $S_1(a)$：$0\le x_1\le a$ 之测度为长度 $V(S_1(a))=a$；

二维单形 $S_2(a)$：$\begin{cases}0\le x_1+x_2\le a\\ x_1\ge 0, x_2\ge 0\end{cases}$ 之测度为面积



$V(S_2(a))=\frac{1}{2}a^2$；三维单形 $S_3(a)$：$\begin{cases}0\le x_1+x_2+x_3\le a\\ x_1\ge 0, x_2\ge 0\\ x_3\ge 0\end{cases}$

的测度为体积 $V(S_3(a))=\frac{1}{6}a^3=\frac{a^3}{3!}$.



(2°) 可以证明：$V(S_n(a))=\frac{a^n}{n!}$.

证：$V(S_n(a))=\int\int\cdots\int_{S_n(a)} 1\,dx_1dx_2\cdots dx_n$

作可逆线性变换 $\begin{pmatrix}x_1\\x_2\\ \vdots\\x_n\end{pmatrix}=\begin{pmatrix}a & & 0\\ & a & \\ 0 & & \ddots & a\end{pmatrix}\begin{pmatrix}u_1\\u_2\\ \vdots\\u_n\end{pmatrix}$. 则 $dx_1dx_2\cdots dx_n=$

$\left|\frac{\partial(x_1,x_2,\cdots x_n)}{\partial(u_1,u_2,\cdots u_n)}\right|du_1du_2\cdots du_n=\begin{vmatrix}a & & 0\\ & a & \\ 0 & & \ddots & a\end{vmatrix}du_1du_2\cdots du_n$

$=a^n du_1du_2\cdots du_n$. (1)

此时，$S_n(a) \xleftrightarrow{\text{一一对应}} S_n(1)$

$$V(S_n(a)) = a^n \underset{u_1+u_2+\cdots+u_n\le 1}{\int\int\cdots\int} 1\,du_1du_2\cdots du_n = a^n V(S_n(1))$$

而 $$V(S_n(1)) = \int_0^1 \Big(\underset{u_2+u_3+\cdots+u_n\le 1-u_1}{\int\int\cdots\int} 1\,du_2du_3\cdots du_n\Big)du_1$$

$$= \int_0^1 (1-u_1)^{n-1} V(S_{n-1}(1))\,du_1 = \frac{V(S_{n-1}(1))}{-1}\int_0^1 (1-u_1)^{n-1}\,d(1-u_1)$$

$$= \frac{V(S_{n-1}(1))}{n} = \frac{1}{n(n-1)}V(S_{n-2}(1)) = \cdots = \frac{1}{n(n-1)(n-2)\times\cdots\times 4\times 3\times 2}\times 1 = \frac{1}{n!}$$

故 $V(S_n(a)) = \frac{a^n}{n!}$，$\forall n\in N^*$　　(※1)

(三) 设 $B_n(a): x_1^2+x_2^2+\cdots+x_n^2 \le a^2$ $(a>0)$ 表示半径为 a 的 n 维球体(ball)，当 $a>0$ 时，称 $B_n(a)$ 的 n 维体积记为 $V(B_n(a))$.

解：(1°) $V(B_1(a)) = 2a$ (长度)，$V(B_2(a)) = \pi a^2$ (面积)；

$V(B_3(a)) = \frac{4}{3}\pi a^3$ (体积)

(2°) 可以证明：$V(B_n(a)) = (\sqrt{\pi}a)^n / \Gamma(\frac{n}{2}+1)$，$\forall n\in N^*$.　(※2)

其中 $\Gamma(s) = \int_0^{+\infty} t^{s-1}e^{-t}\,dt$ $(s>0)$ 为伽玛函数 (参§13.5.1)

证: $\because V(B_n(a)) = \underset{(\frac{x_1}{a})^2+(\frac{x_2}{a})^2+\cdots+(\frac{x_n}{a})^2\le 1}{\int\int\cdots\int} 1\,dx_1dx_2\cdots dx_n$

$\because$ 当作可逆线性变换 $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a & & 0 \\ & a & \\ 0 & & \ddots & a \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ 时,

$dx_1dx_2\cdots dx_n = a^n du_1du_2\cdots du_n$ 且 $u_1^2+u_2^2+\cdots+u_n^2 \le 1$

$$V(B_n(a)) = a^n \underset{u_1^2+\cdots+u_n^2\le 1}{\int\int\cdots\int} du_1du_2\cdots du_n = a^n V(B_n(1))$$

而 $V(B_n(1)) = \underset{u_1^2+u_2^2\le 1}{\int\int} \left(\underset{u_3^2+\cdots+u_n^2\le 1-u_1^2-u_2^2}{\int\int\cdots\int} 1\,du_3du_4\cdots du_n \right) du_1du_2$

$$= \underset{u_1^2+u_2^2\le 1}{\int\int} (1-u_1^2-u_2^2)^{\frac{n-2}{2}} V(B_{n-2}(1))\,du_1du_2 \xlongequal[u_2=r\sin\theta]{u_1=r\cos\theta}$$

$$V(B_{n-2}(1)) \int_0^{2\pi}\left(\int_0^1 (1-r^2)^{\frac{n}{2}-1} r\,dr\right)d\theta = \frac{2\pi}{n} V(B_{n-2}(1))$$

从而 $V(B_{2n}(1)) = \frac{2\pi}{2n} V(B_{2n-2}(1)) = \frac{2\pi}{2n}\frac{2\pi}{2n-2} V(B_{2n-4}(1)) = \cdots$

$= \frac{2\pi}{2n}\cdot\frac{2\pi}{2n-2}\cdot\frac{2\pi}{2n-4}\cdots\frac{2\pi}{4}\cdot V(B_2(1))$, 且 $V(B_2(1)) = \pi$.

故 $V(B_{2n}(1)) = \frac{\pi^n}{n!}$, $\forall n\in\mathbb{N}^* \Rightarrow V(B_{2n}(a)) = \frac{a^{2n}\pi^n}{n!}$. (★3)

(3).

$$V(B_{2n+1}(1))=\frac{2\pi}{2n+1}\cdot\frac{2\pi}{2n-3}\frac{2\pi}{2n-5}\cdots\frac{2\pi}{5}\frac{2\pi}{3}V(B_1(1))$$

且 $V(B_1(1))=2$，$\therefore V(B_{2n+1}(1))=2^n\pi^{n+1}/(2n+1)!! \Rightarrow$

$V(B_{2n+1}(a))=a^{2n+1}V(B_{2n+1}(1))=a^{2n+1}2^n\pi^{n+1}/(2n+1)!!$，$\forall n\in N^*$，(式4)

(式3)，(式4)可用伽玛函数 $\Gamma(s)$ 统一表示为：

$$V(B_n(a))=(\sqrt{\pi}a)^n/\Gamma(\frac{n}{2}+1),\ \forall n\in N^*$$

利用(式4)，(式3)可知：

$V(B_4(a))\overset{n=2}{=\!=}\pi^2a^4/2$；$V(B_5(a))\overset{n=2}{=\!=}a^5\cdot2^2\cdot\pi^2/5!!=\frac{8}{15}\pi^2a^5$；

$V(B_6(a))\overset{n=3}{=\!=}a^6\pi^3/3!=\frac{1}{6}\pi^3a^6$；$\cdots$，$V(B_{10}(a))=\frac{\pi^5a^{10}}{120}$.

(三) 用正交变换 $\begin{pmatrix}u_1\\u_2\\\vdots\\u_n\end{pmatrix}=A\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}$ 化简 n 重积分：

$$I=\underset{x_1^2+x_2^2+\cdots+x_n^2\le1}{\int\int\cdots\int}f(a_1x_1+a_2x_2+\cdots+a_nx_n)dx_1dx_2\cdots dx_n,$$

其中 $(a_1,a_2,\cdots a_n)\neq\theta$ 是常向量，$f\in C$.

例1. 化简 $I=\underset{x_1^2+x_2^2+x_3^2+x_4^2\le1}{\int\int\int\int}f(a_1x_1+a_2x_2+a_3x_3+a_4x_4)dx_1dx_2dx_3dx_4$

解：令 $\lambda=(a_1^2+a_2^2+a_3^2+a_4^2)^{\frac{1}{2}}$，则 $\lambda>0$. 设4阶方阵 A

$$=\begin{pmatrix}\frac{a_1}{\lambda} & \frac{a_2}{\lambda} & \frac{a_3}{\lambda} & \frac{a_4}{\lambda}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ \cdots & & & \\ a_{41} & a_{42} & a_{43} & a_{44}\end{pmatrix}$$ 为正交阵，即满足 $AA^T=E=A^TA$

作正交变换：$\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix}=A\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$ 则 $\lambda u_1=a_1x_1+a_2x_2+a_3x_3+a_4x_4$

且 $dx_1dx_2dx_3dx_4=\left|\frac{\partial(x_1\ x_2\ x_3\ x_4)}{\partial(u_1,u_2,u_3,u_4)}\right|du_1du_2du_3du_4=\frac{du_1du_2du_3du_4}{|\pm1|}$

$u_1^2+u_2^2+u_3^2+u_4^2=(u_1\ u_2\ u_3\ u_4)\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix}=(x_1\ x_2\ x_3\ x_4)A^TA\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=x_1^2+x_2^2+x_3^2+x_4^2\le1$.

$$I=\iiiint\limits_{u_1^2+u_2^2+u_3^2+u_4^2\le1}f(\lambda u_1)du_1du_2du_3du_4=\int_{-1}^{1}f(\lambda u_1)\Big(\iiint\limits_{u_2^2+u_3^2+u_4^2\le1-u_1^2}1\,du_2du_3du_4\Big)du_1$$

$$=\int_{-1}^{1}f(\lambda u_1)\frac{4}{3}\pi(1-u_1^2)^{\frac{3}{2}}du_1=\begin{cases}0, & \text{若}f\text{为奇函数}\\ \frac{8\pi}{3}\int_0^1 f(\lambda u_1)(1-u_1^2)^{\frac{3}{2}}du_1, & f\text{为偶}\\ \text{其它.} & \end{cases}$$

例2. 化简 $I=\iint\cdots\int\limits_{x_1^2+x_2^2+x_3^2+x_4^2+x_5^2\le1}f(a_1x_1+a_2x_2+a_3x_3+a_4x_4+a_5x_5)dx_1dx_2dx_3dx_4dx_5$

解：令 $\lambda=(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2)^{\frac{1}{2}}$，则 $\lambda>0$. 取5阶正交阵

$A=\begin{pmatrix}\frac{a_1}{\lambda} & \frac{a_2}{\lambda} & \frac{a_3}{\lambda} & \frac{a_4}{\lambda} & \frac{a_5}{\lambda}\\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25}\\ \cdots & & & & \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55}\end{pmatrix}$，作正交变换 $\begin{pmatrix}u_1\\u_2\\u_3\\u_4\\u_5\end{pmatrix}=A\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}$

则 $dx_1dx_2dx_3dx_4dx_5 = \dfrac{du_1du_2du_3du_4du_5}{\left|\dfrac{\partial(u_1,u_2,u_3,u_4,u_5)}{\partial(x_1,x_2,x_3,x_4,x_5)}\right|} = \dfrac{du_1du_2du_3du_4du_5}{||A||}$

$= du_1du_2du_3du_4du_5$ 且 $u_1^2+u_2^2+u_3^2+u_4^2+u_5^2 = x_1^2+x_2^2+x_3^2+x_4^2+x_5^2 \le 1$

$a_1x_1+a_2x_2+a_3x_3+a_4x_4+a_5x_5 = \lambda u_1$，$\Rightarrow$

$$I = \iiiint\!\!\int_{u_1^2+u_2^2+u_3^2+u_4^2+u_5^2\le 1} f(\lambda u_1)\,du_1du_2du_3du_4du_5$$

$$= \int_{-1}^{1} f(\lambda u_1)du_1 \iiiint_{u_2^2+u_3^2+u_4^2+u_5^2\le (1-u_1^2)^{\frac{2}{2}}} 1\,du_2du_3du_4du_5$$

$$= \int_{-1}^{1} f(\lambda u_1)\,\frac{\pi^2\left[(1-u_1^2)^{\frac{1}{2}}\right]^4}{2}du_1 = \frac{\pi^2}{2}\int_{-1}^{1}(1-u_1^2)^2 f(\lambda u_1)du_1.$$

余类推。

(四) 作业：

(1). 推导半径为 a 的 n 维球体 $B_n(a)$ 的体积公式 $(a>0)$;

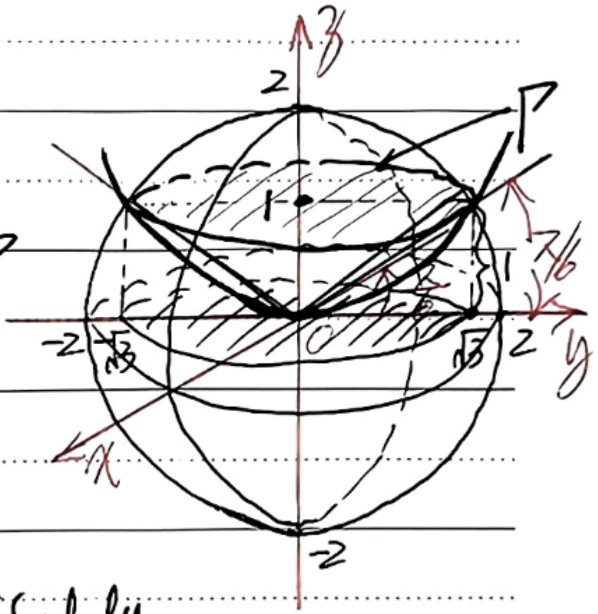
(2) 化简：$I = \iint\cdots\int_{x_1^2+x_2^2+\cdots+x_6^2\le 1} f(a_1x_1+a_2x_2+a_3x_3+a_4x_4+a_5x_5+a_6x_6)\,dx_1\cdots dx_6$

$(a_1,a_2,\cdots,a_6)\ne 0$, $f\in C$

(3) ex10.4/1; ch10综./3.

第24讲（附页）

(一)、计算三重积分：$I=\iiint\limits_{\Omega} z\,dV$，其中，$\Omega$是由球面$\Sigma_1$：$x^2+y^2+z^2=4$与旋转抛物面$\Sigma_2$：$x^2+y^2=3z$围成的区域。



解：Σ_1与Σ_2的交线Γ为：从

$x^2+y^2=4-z^2=3z \Rightarrow z=1$ 即交线Γ

的方程为 $\begin{cases} x^2+y^2=(\sqrt{3})^2 \\ z=1 \end{cases}$

解法(一)："先二后一"法：

$$I=\int_0^1 z\,dz \iint\limits_{x^2+y^2\le 3z} 1\,dxdy+\int_1^2 z\,dz \iint\limits_{x^2+y^2\le 4-z^2} 1\,dxdy$$

$$=\int_0^1 z\pi\cdot 3z\,dz+\int_1^2 z\pi(4-z^2)\,dz=\pi+\frac{9}{4}\pi=\frac{13}{4}\pi;$$

解法(二)："先一后二"法：

$$I=\iint\limits_{x^2+y^2\le 3}\left(\int_{\frac{1}{3}(x^2+y^2)}^{\sqrt{4-x^2-y^2}} z\,dz\right)dxdy=\frac{1}{2}\iint\limits_{x^2+y^2\le 3}\left(z^2\Big|_{\frac{1}{3}(x^2+y^2)}^{\sqrt{4-x^2-y^2}}\right)dxdy$$

$$=\frac{1}{2}\iint\limits_{x^2+y^2\le 3}\left[(4-x^2-y^2)-\frac{1}{9}(x^2+y^2)^2\right]dxdy \xlongequal[y=r\sin\theta]{令x=r\cos\theta}$$

(1),

$$=\frac{1}{2}\int_0^{2\pi}d\theta\int_0^{\sqrt{3}}\left[4-r^2-\frac{1}{9}(r^2)^2\right]r\,dr$$

$$=\frac{1}{2}\int_0^{2\pi}d\theta\times\left(\int_0^{\sqrt{3}}\left(4r-r^3-\frac{1}{9}r^5\right)dr\right)=\pi\cdot\frac{13}{4};$$

解法(三)：柱坐标变换法：令 $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\\ z=z\end{cases}$，则

$dV=dxdydz=\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right|drd\theta dz=r\,drd\theta dz$，且 $I=$

$$=\int_0^{2\pi}d\theta\int_0^{\sqrt{3}}dr\int_{\frac{1}{3}r^2}^{\sqrt{4-r^2}}z\,r\,dz$$

$$=\int_0^{2\pi}d\theta\int_0^{\sqrt{3}}r\,\frac{1}{2}\left(4-r^2-\frac{1}{9}r^4\right)dr$$

$$=\int_0^{2\pi}d\theta\int_0^{\sqrt{3}}\frac{1}{2}\left(4r-r^3-\frac{1}{9}r^5\right)dr=\pi\cdot\frac{13}{4};$$

解法(四)：球坐标变换法：令 $\begin{cases}x=r\sin\theta\cos\varphi\\ y=r\sin\theta\sin\varphi\\ z=r\cos\theta\end{cases}$

则 $dV=dxdydz=\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right|drd\theta d\varphi=r^2\sin\theta\,drd\theta d\varphi$ 且

$$I=\int_0^{2\pi}d\varphi\int_0^{\frac{\pi}{3}}d\theta\int_0^{2}(r\cos\theta)\,r^2\sin\theta\,dr+\int_0^{2\pi}d\varphi\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}d\theta\int_0^{\frac{3\cos\theta}{\sin^2\theta}}r\cos\theta\,r^2\sin\theta\,dr$$

$$=\int_0^{2\pi}1\,d\varphi\left(\int_0^{\frac{\pi}{3}}\cos\theta\sin\theta\,d\theta\right)\left(\int_0^2 r^3dr\right)+\int_0^{2\pi}1\,d\varphi\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\frac{1}{4}\left(\frac{3\cos\theta}{\sin^2\theta}\right)^4\sin\theta\cos\theta\,d\theta$$

$$=2\pi\times\frac{3}{8}\times4+2\pi\times\frac{81}{4}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\frac{\cos^5\theta}{\sin^7\theta}d\theta$$

$$=3\pi+\frac{\pi}{2}\times81\left(-\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\cot^5\theta\,d(\cot\theta)\right)$$

$= 3\pi - \frac{81}{12}\pi\cot^6\theta \Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} = 3\pi + \frac{\pi}{4} = \frac{13}{4}\pi.$

(三) 求椭球体 $\Omega: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$ 的体积 $V(\Omega)$ 的

五种方法:

I) "先一后二"法:

$$V(\Omega)=\iiint_\Omega 1\,dV=\iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1}\left(\int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} 1\,dz\right)dxdy$$

$$=\iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1} 2c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dxdy \xrightarrow[x=ar\cos\theta,\ y=br\sin\theta]{\text{广义极坐标变换}}$$

$$=2c\int_0^{2\pi}d\theta\int_0^1\sqrt{1-r^2}\,abr\,dr=2abc\left(\int_0^{2\pi}d\theta\right)\left(\int_0^1 r\sqrt{1-r^2}\,dr\right)$$

$$=2abc\times 2\pi\times\frac{1}{3}=\frac{4}{3}\pi abc;$$

II) "先二后一"法:

$$V(\Omega)=\iiint_\Omega 1\,dV=\int_{-c}^{c}1\,dz\iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1-\frac{z^2}{c^2}}1\,dxdy$$

而 $$\iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1-\frac{z^2}{c^2}}1\,dxdy=\pi\left(a\sqrt{1-\frac{z^2}{c^2}}\right)\left(b\sqrt{1-\frac{z^2}{c^2}}\right)=\pi ab\left(1-\frac{z^2}{c^2}\right)$$

$$\therefore V(\Omega)=\int_{-c}^{c}\pi ab\left(1-\frac{z^2}{c^2}\right)dz=2\int_0^c\pi ab\left(1-\frac{z^2}{c^2}\right)dz=\frac{4}{3}\pi abc;$$

(3).

(III) "广义球坐标变换法" 令 $\begin{cases} x = a r\sin\theta\cos\varphi \\ y = b r\sin\theta\sin\varphi \\ z = c r\cos\theta \end{cases}$，则

$$dV = dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| dr\,d\theta\,d\varphi = abc\,r^2\sin\theta\,dr\,d\theta\,d\varphi$$，且

$$I = V(\Omega) = \iiint_\Omega 1\,dV = \int_0^{2\pi} 1\,d\varphi \int_0^{\pi} 1\,d\theta \int_0^1 abc\,r^2\sin\theta\,dr$$

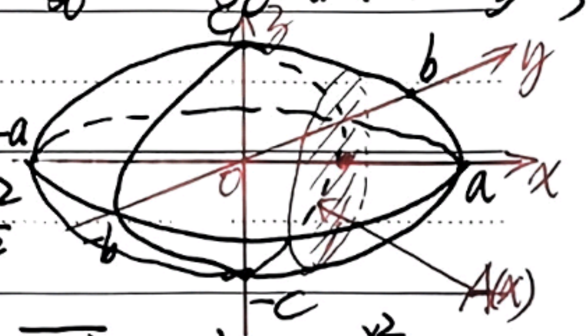
$$= \int_0^{2\pi} 1\,d\varphi \times \left(\int_0^{\pi}\sin\theta\,d\theta\right)\left(\int_0^1 abc\,r^2\,dr\right) = 2\pi\cdot 2\cdot\frac{abc}{3} = \frac{4}{3}\pi abc;$$

(IV) 利用曲顶：$z = c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$ 且底面为 $D: \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$ 的曲顶柱体体积法：即"二重积分几何意义法"：

$$V(\Omega) = 2\iint_D c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy \xrightarrow[y = b r\sin\theta]{\text{令 } x = a r\cos\theta}$$

$$ab\cdot 2\int_0^{2\pi} 1\,d\theta \int_0^1 c\sqrt{1-r^2}\,r\,dr = 2abc\left(\int_0^{2\pi} 1\,d\theta \times \int_0^1 r\sqrt{1-r^2}\,dr\right) = \frac{4\pi abc}{3};$$

(V) "定积分法"，当 $x\in[-a,a]$ 时，对应的截面区域为 $D_x: \frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1-\frac{x^2}{a^2}$



D_x 的面积为 $A(x) = \pi\left(b\sqrt{1-\frac{x^2}{a^2}}\right)\left(c\sqrt{1-\frac{x^2}{a^2}}\right) = \pi bc\left(1-\frac{x^2}{a^2}\right)$.

即 $V(\Omega) = \int_{-a}^{a} A(x)\,dx = \int_{-a}^{a}\pi bc\left(1-\frac{x^2}{a^2}\right)dx = \frac{4}{3}\pi abc.$

椭球体Ω属于"截面积为已知函数A(x)的立体"。故V(Ω)可用一重积分、二重积分、三重积分分别来计算。

(4).