第9讲：复合(隐)函数微分法

(一) 复合函数(composition)的微分法：

Th1：设 $z=f(u,v)$ 在区域 D 中可微，且 $\begin{cases} u=g(x,y) \\ v=h(x,y) \end{cases}$ 都在区域 E 中可微．当复合 $f(g(x,y),h(x,y))$ 有意义时，z 通过中间变量 u,v，成为 x,y 的二元复合函数，且有求偏导数的链式法则如下：

$$\begin{cases} \dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial x} & (※1) \\[2ex] \dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y}+\dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y} & (※2). \end{cases}$$

且不论 u,v 是 $f(u,v)$ 的自变量，还是作为复合函数 $f(g(x,y),h(x,y))$ 的中间变量，总有：

$$dz=\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv \qquad (※3)$$

(※3)称为全微分的一阶形式不变性。

证(※1)：固定 y，让 x 有增量 Δx，则 $\begin{cases} \Delta u_x=g(x+\Delta x,y)-g(x,y) \\ \Delta v_x=h(x+\Delta x,y)-h(x,y) \end{cases}$

x的变化，通过u,v使z产生了变化：

$$\Delta z_x = f(u+\Delta u_x, v+\Delta v_x) - f(u,v) = \frac{\partial z}{\partial u}\Delta u_x + \frac{\partial z}{\partial v}\Delta v_x + o(\rho)$$

$\rho = \sqrt{(\Delta u_x)^2+(\Delta v_x)^2}$，从而有：

$$\frac{\Delta z_x}{\Delta x} = \frac{\partial z}{\partial u}\frac{\Delta u_x}{\Delta x} + \frac{\partial z}{\partial v}\frac{\Delta v_x}{\Delta x} + \frac{o(\rho)}{\Delta x} \quad 且 \begin{cases} \lim\limits_{\Delta x\to 0}\dfrac{\Delta u_x}{\Delta x} = \dfrac{\partial u}{\partial x}, \\ \lim\limits_{\Delta x\to 0}\dfrac{\Delta v_x}{\Delta x} = \dfrac{\partial v}{\partial x}, \end{cases}$$

$$\frac{o(\rho)}{\Delta x} = \frac{o(\rho)}{\rho}\cdot\frac{\rho}{\Delta x} = \frac{o(\rho)}{\rho}\sqrt{\left(\frac{\Delta u_x}{\Delta x}\right)^2+\left(\frac{\Delta v_x}{\Delta x}\right)^2} \to 0\cdot\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2} = 0$$

从而 $\dfrac{\partial z}{\partial x} = \lim\limits_{\Delta x\to 0}\dfrac{\Delta z_x}{\Delta x} = \dfrac{\partial z}{\partial u}\left(\lim\limits_{\Delta x\to 0}\dfrac{\Delta u_x}{\Delta x}\right) + \dfrac{\partial z}{\partial v}\left(\lim\limits_{\Delta x\to 0}\dfrac{\Delta v_x}{\Delta x}\right) + 0$

$$= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

同理，从 $\Delta z_y = f(u+\Delta u_y, v+\Delta v_y) - f(u,v) = \dfrac{\partial z}{\partial u}\Delta u_y + \dfrac{\partial z}{\partial v}\Delta v_y + o(\rho)$

$\rho = \sqrt{(\Delta u_y)^2+(\Delta v_y)^2}$ 可证得：$\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y}$.

证(2)：当u,v作$f(u,v)$的自变量时，$\because z=f(u,v)$可微.

显然有：$dz = \dfrac{\partial z}{\partial u}du + \dfrac{\partial z}{\partial v}dv$.

当u,v作为复合函数$f(g(x,y),h(x,y))$的中间变量时，

从 $z=f(g(x,y),h(x,y)) \Longrightarrow$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)dy$$

$$= \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right) = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv$$

全微分的一阶形式不变性成立。 (2).

(三) 隐函数(implicit function)求微分法：

例1. 方程：$3x+4y-5z+7=0$ 既可确定函数：

$z=\frac{3}{5}x+\frac{4}{5}y+\frac{7}{5}$，也可确定函数 $y=-\frac{3}{4}x+\frac{5}{4}z-\frac{7}{4}$ 及

函数：$x=-\frac{4}{3}y+\frac{5}{3}z-\frac{7}{3}$，利用：$\begin{cases}\frac{\partial z}{\partial x}=\frac{3}{5}, \\ \frac{\partial z}{\partial y}=\frac{4}{5},\end{cases}$ $\begin{cases}\frac{\partial y}{\partial z}=\frac{5}{4}, \\ \frac{\partial y}{\partial x}=-\frac{3}{4},\end{cases}$

$\begin{cases}\frac{\partial x}{\partial y}=-\frac{4}{3} \\ \frac{\partial x}{\partial z}=\frac{5}{3}.\end{cases}$ 可得：$\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z}=\frac{3}{5}\times(-\frac{4}{3})\times\frac{5}{4}=-1.$

同理：$\frac{\partial x}{\partial z}\cdot\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial x}=\frac{5}{3}\times\frac{4}{5}\times(-\frac{3}{4})=-1.$

上述的三个二元函数，都是方程 $3x+4y-5z+7\triangleq F(x,y,z)=0$

所确定的隐函数。

Th2：设方程 $F(x,y)=0$ 满足：$\begin{cases}① F(x,y)\in C^1(D), D\text{为区域} \\ ② F(x_0,y_0)=F(M_0)=0, M_0\in D. \\ ③ F_y(M_0)=F_y(x_0,y_0)\neq 0,\end{cases}$

则方程 $F(x,y)=0$ 可在点 M_0 的某个邻域 $U(M_0,\delta)$ 中唯一

确定隐函数：$y=g(x)$. 且 $\begin{cases}① g(x_0)=y_0 \\ ② \frac{dy}{dx}=g'(x)=-\frac{F_x(x,y)}{F_y(x,y)}\in C\end{cases}$

Th3：设方程 $F(x,y,z)=0$ 满足 ① $F(x,y,z)\in C^1(D)$，D为区域

② $F(M_0)=F(x_0,y_0,z_0)=0$，$M_0\in D$

③ $F'_z(M_0)\neq 0$，

则方程 $F(x,y,z)=0$ 在 M_0 的某个δ邻域 $U(M_0,\delta)$ 中可唯一确定隐函数：$z=\varphi(x,y)$ 且 ① $\varphi(x_0,y_0)=z_0$

② $\frac{\partial z}{\partial x}=-\frac{F'_x(x,y,z)}{F'_z(x,y,z)}$，$\frac{\partial z}{\partial y}=-\frac{F'_y(x,y,z)}{F'_z(x,y,z)}$

证Th2：不妨设 $F'_y(x_0,y_0)=F'_y(M_0)>0$，则 $F(x_0,y)$ 在 y_0 的附近严格单调增，即在 $M_0(x_0,y_0)$ 附近形成了一条唯一确定的单值单连续曲线，设这曲线的表达式为 $y=\varphi(x)$，$(x,y)\in U(M_0,\delta)$，则 $y=\varphi(x)$ 即为所求的隐函数。

显然 $y=\varphi(x)$ 穿过点 $M_0(x_0,y_0)$，即 $\varphi(x_0)=y_0$。且从 $F(x,\varphi(x))=0$ 两边对x求导得：$F'_x\cdot 1+F'_y\cdot\frac{d\varphi(x)}{dx}\equiv 0\Rightarrow\frac{d\varphi(x)}{dx}=\varphi'(x)=\frac{dy}{dx}$

$=-\frac{F'_x(x,y)}{F'_y(x,y)}$，从 $F\in C^1(D)$ 知，$\varphi'(x)$ 是连续函数。

值得注意的是，这些隐函数 $y=\varphi(x)$ 总是难以求出，实际问题中未必能求出来！但隐函数的导数或偏导数

(4).

是否能够从已知的方程 $F(x,y)=0$ 或 $F(x,y,z)=0$ 中求出来!

例如：若已知 $z=g(x,y)$ 是方程 $F(x,y,z)=0$ 确定的隐函数，则 $F(x,y,g(x,y))=0$，两边对 x,y 分别求导：

$$\begin{cases} F'_x\cdot 1+F'_z\cdot g'_x(x,y)=0 \\ F'_y\cdot 1+F'_z\cdot g'_y(x,y)=0 \end{cases} \Rightarrow \begin{cases} g'_x(x,y)=\dfrac{\partial z}{\partial x}=-\dfrac{F'_x(x,y,z)}{F'_z(x,y,z)} \\ g'_y(x,y)=\dfrac{\partial z}{\partial y}=-\dfrac{F'_y(x,y,z)}{F'_z(x,y,z)} \end{cases}$$

三) 例题：

(1) 证明：$u=\dfrac{1}{r}$，$r=\sqrt{x^2+y^2+z^2}>0$ 满足 Laplace 方程：

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0,\ \forall (x,y,z)\neq(0,0,0)$$

(2) 证明：$u=\dfrac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}$ $(x>0,t>0,a>0$ 常数$)$

满足热传导方程：$\dfrac{\partial u}{\partial t}=a^2\dfrac{\partial^2 u}{\partial x^2}$.

(3) 设 $\varphi,\psi\in C^2(I)$. 证明：$u=\varphi(x-at)+\psi(x+at)$ 满足

波动方程：$\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$ $(a>0$ 常数$, t>0, x\in R)$

证(1) $\because \dfrac{\partial u}{\partial x}=\dfrac{du}{dr}\cdot\dfrac{\partial r}{\partial x}=-\dfrac{1}{r^2}\cdot\dfrac{x}{r}=\dfrac{-x}{r^3}$

$$\therefore \frac{\partial^2 u}{\partial x^2} = -\left(\frac{x}{r^3}\right)'_x = -\frac{1\cdot r^3-3r^2\frac{x}{r}\cdot x}{r^6} = -\frac{r^2-3x^2}{r^5}$$

由 $u=\frac{1}{\sqrt{x^2+y^2+z^2}}$ 的对称性知：$\begin{cases} \frac{\partial^2 u}{\partial y^2} = -\frac{r^2-3y^2}{r^5} \\ \frac{\partial^2 u}{\partial z^2} = -\frac{r^2-3z^2}{r^5} \end{cases}$

故 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = -\frac{3r^2-3(x^2+y^2+z^2)}{r^5} = -\frac{3r^2-3r^2}{r^5} \equiv 0.$

证(2)：$\frac{\partial u}{\partial t} = \frac{(t^{-\frac{1}{2}})'_t}{2a\sqrt{\pi}} e^{-\frac{x^2}{4a^2t}} + \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2t}} \left(-\frac{x^2}{4a^2t}\right)'_t$

$$= \frac{1}{4a\sqrt{\pi t}\, t} e^{-\frac{x^2}{4a^2t}} \left(\frac{x^2}{2a^2t}-1\right),$$

而 $\frac{\partial u}{\partial x} = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2t}} \left(-\frac{x^2}{4a^2t}\right)'_x = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2t}} \left(-\frac{x}{2a^2t}\right) \Rightarrow$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2a\sqrt{\pi t}} \left[e^{-\frac{x^2}{4a^2t}} \left(-\frac{x}{2a^2t}\right)^2 + e^{-\frac{x^2}{4a^2t}} \left(-\frac{1}{2a^2t}\right)\right]$$

$$= \frac{1}{4a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2t}} \left(\frac{x^2}{2a^4t}-\frac{1}{a^2}\right) \Rightarrow$$

$a^2\frac{\partial^2 u}{\partial x^2} = \frac{1}{4a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2t}} \left(\frac{x^2}{2a^2t}-1\right) = \frac{\partial u}{\partial t}$，$\forall t>0, x>0$.

证(3)：令 $\begin{cases} v=x-at \\ w=x+at \end{cases}$，则 $u=\varphi(v)+\psi(w)$ 且 $\begin{cases} \frac{\partial v}{\partial x}=1 \\ \frac{\partial w}{\partial x}=1 \end{cases}$，

$\begin{cases} \frac{\partial v}{\partial t}=-a \\ \frac{\partial w}{\partial t}=a \end{cases}$ $\Rightarrow \frac{\partial u}{\partial x} = \varphi'(v)\frac{\partial v}{\partial x}+\psi'(w)\frac{\partial w}{\partial x} = \varphi'(v)\cdot 1+\psi'(w)\cdot 1$

$$\frac{\partial^2 u}{\partial x^2} = \varphi''(v)\cdot 1^2+\psi''(w)\cdot 1^2 = \varphi''(v)+\psi''(w).$$

而 $\frac{\partial u}{\partial t}=\varphi'(v)\frac{\partial v}{\partial t}+\psi'(w)\frac{\partial w}{\partial t}=\varphi'(v)(-a)+\psi'(w)a$

$$\frac{\partial^2 u}{\partial t^2}=\varphi''(v)(-a)^2+\psi''(w)a^2=a^2(\varphi''(v)+\psi''(w))=a^2\frac{\partial^2 u}{\partial x^2}.$$

(4). 球面方程 $x^2+y^2+z^2=a^2$ ($a>0$, 常数) 在第一卦限可确定三个隐函数：$x=\sqrt{a^2-y^2-z^2}$；$y=\sqrt{a^2-x^2-z^2}$；$z=\sqrt{a^2-x^2-y^2}$，证明：$\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z}\cdot\frac{\partial z}{\partial x}\equiv-1$.

证：$\because \frac{\partial x}{\partial y}=-\frac{2y}{2\sqrt{a^2-y^2-z^2}}=-\frac{y}{x}$；$\frac{\partial y}{\partial z}=-\frac{2z}{2\sqrt{a^2-x^2-z^2}}=-\frac{z}{y}$；

$\frac{\partial z}{\partial x}=-\frac{2x}{2\sqrt{a^2-x^2-y^2}}=-\frac{x}{z}$，$(x>0, y>0, z>0)$

$\therefore \frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z}\cdot\frac{\partial z}{\partial x}=\left(\frac{-y}{x}\right)\left(\frac{-z}{y}\right)\left(\frac{-x}{z}\right)\equiv-1$. $\forall x>0, y>0, z>0$ 且 $x^2+y^2+z^2=a^2$.

(5). 设 $F(x,y)\in C^2(D)$, D是区域，由函数 $y=\varphi(x)$ 由方程 $F(x,y)=0$ 确定，证明：

$$\varphi''(x)=\frac{d^2y}{dx^2}=\frac{\frac{\partial^2 F}{\partial x^2}\left(\frac{\partial F}{\partial y}\right)^2-2\frac{\partial^2 F}{\partial x\partial y}\frac{\partial F}{\partial x}\frac{\partial F}{\partial y}+\frac{\partial^2 F}{\partial y^2}\left(\frac{\partial F}{\partial x}\right)^2}{\left(\frac{\partial F}{\partial y}\right)^3}\ (-1)$$

证 1°) $\because \varphi'(x)=\frac{dy}{dx}=-\frac{F'_x(x,y)}{F'_y(x,y)}\triangleq y'_x$.

2°) $\varphi''(x)=-\left(\frac{F'_x(x,y)}{F'_y(x,y)}\right)'_x=-\frac{(F'_x(x,y))'_x F'_y(x,y)-(F'_y(x,y))'_x F'_x(x,y)}{(F'_y(x,y))^2}$

$\square$.

$$= \frac{(F''_{xx}\cdot 1 + F''_{xy}\cdot y'_x)F'_y - (F''_{yx}\cdot 1 + F''_{yy}\cdot y'_x)F'_x}{(F'_y)^2} (-1)$$

利用 $y'_x = -F'_x/F'_y$ $\frac{(F''_{xx} + F''_{xy}(-\frac{F'_x}{F'_y}))F'_y - (F''_{xy} + F''_{yy}(-\frac{F'_x}{F'_y}))F'_x}{(F'_y)^2} (-1)$

$$= \frac{F''_{xx}(F'_y)^2 - F''_{xy}F'_xF'_y - F''_{xy}F'_xF'_y + F''_{yy}(F'_x)^2}{(F'_y)^3} (-1)$$

$$= \frac{\frac{\partial^2 F}{\partial x^2}(\frac{\partial F}{\partial y})^2 - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial F}{\partial x}\frac{\partial F}{\partial y} + \frac{\partial^2 F}{\partial y^2}(\frac{\partial F}{\partial x})^2}{(\frac{\partial F}{\partial y})^3} (-1)$$

(四) 作业：

ex 9.2：20/(2),(3),(4)；25；28；32；

ex 9.3：1/(1)；2/(2),(5)；4/(1).