第四周作业答案

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习题 9.2

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$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=yz\\\frac{\partial u}{\partial y}&=xz\\\frac{\partial u}{\partial z}&=xy\end{aligned}\right\}\Longrightarrow\nabla u=(yz,xz,xy)$$

于是所求方向导数为

$$\nabla u(1,2,-1)\cdot\frac{\mathbf{l}}{|\mathbf{l}|}=-\frac{3}{\sqrt{11}}$$

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将该圆参数化为 $\mathbf{r}(\theta)=(1+\cos\theta,\sin\theta)$，则逆时针为正向。此时 P 点对应 $\theta=\frac{2\pi}{3}$。因此，P 处的单位且向量为

$$\mathbf{t}=\left.\frac{r'(\theta)}{|r'(\theta)|}\right|_{\theta=\frac{2\pi}{3}}=\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$$

从而方向导数为

$$\nabla z\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)\cdot\mathbf{t}=\left.\left(-\frac{y}{x^2+y^2},\frac{x}{x^2+y^2}\right)\right|_{(x,y)=(\frac{1}{2},\frac{\sqrt{3}}{2})}\cdot\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)=\frac{1}{2}$$

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$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=2x+y+3\\\frac{\partial u}{\partial y}&=4y+x-2\\\frac{\partial u}{\partial z}&=6z-6\end{aligned}\right\}\Longrightarrow\nabla u=(2x+y+3,4y+x-2,6z-6)\Longrightarrow\nabla u(1,1,-1)=(6,3,-12)$$

于是由 Cauchy 不等式，对于单位向量 (a,b,c)

$$\nabla u(1,1,-1)\cdot(a,b,c)\le\sqrt{(36+9+144)(a^2+b^2+c^2)}=3\sqrt{21}$$

即最大方向导数为 $3\sqrt{21}$，此时

$$(a,b,c)=\left(\frac{2}{\sqrt{21}},\frac{1}{\sqrt{21}},-\frac{4}{\sqrt{21}}\right)$$

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$$\nabla\frac{1}{r}=\nabla\frac{1}{x^2+y^2+z^2}=-\frac{2\mathbf{r}}{r^4}$$
$$\nabla\ln r=\nabla\frac{1}{2}\ln(x^2+y^2+z^2)=\frac{\mathbf{r}}{r^2}$$

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证明. 由

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}=(1-\cos x)\frac{\partial u}{\partial \xi}+(1+\cos x)\frac{\partial u}{\partial \eta}$$
$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}=\frac{\partial u}{\partial \xi}-\frac{\partial u}{\partial \eta}$$

知

$$\begin{aligned}
0&=\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}\cos x-\frac{\partial^2 u}{\partial y^2}\sin^2 x-\frac{\partial u}{\partial y}\sin x\\
&=\frac{\partial}{\partial x}\left((1-\cos x)\frac{\partial u}{\partial \xi}+(1+\cos x)\frac{\partial u}{\partial \eta}\right)+2\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}-\frac{\partial u}{\partial \eta}\right)\cos x\\
&-3\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \xi}-\frac{\partial u}{\partial \eta}\right)\sin^2 x-\left(\frac{\partial u}{\partial \xi}-\frac{\partial u}{\partial \eta}\right)\sin x\\
&=\left(\frac{\partial u}{\partial \xi}-\frac{\partial u}{\partial \eta}\right)\sin x+(1-\cos x)\left((1-\cos x)\frac{\partial}{\partial \xi}+(1+\cos x)\frac{\partial}{\partial \eta}\right)\frac{\partial u}{\partial \xi}\\
&+(1+\cos x)\left((1-\cos x)\frac{\partial}{\partial \xi}+(1+\cos x)\frac{\partial}{\partial \eta}\right)\frac{\partial u}{\partial \eta}\\
&+2\cos x(1-\cos x)\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi}-\frac{\partial u}{\partial \eta}\right)+2\cos x(1+\cos x)\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}-\frac{\partial u}{\partial \eta}\right)\\
&-\sin^2 x\left(\frac{\partial}{\partial \xi}-\frac{\partial}{\partial \eta}\right)\left(\frac{\partial u}{\partial \xi}-\frac{\partial u}{\partial \eta}\right)-\left(\frac{\partial u}{\partial \xi}-\frac{\partial u}{\partial \eta}\right)\sin x\\
&=2\frac{\partial^2 u}{\partial \xi\partial y}
\end{aligned}$$

□

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(2)

$$\mathrm{d}u = \frac{\partial u}{\partial x}\,\mathrm{d}x + \frac{\partial u}{\partial y}\,\mathrm{d}y = \left(yf_\xi + \frac{1}{y}f_\eta\right)\mathrm{d}x + \left(xf_\xi - \frac{x}{y^2}f_\eta\right)\mathrm{d}y$$

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$$\mathrm{d}u = \frac{\partial u}{\partial x}\,\mathrm{d}x + \frac{\partial u}{\partial y}\,\mathrm{d}y = 2\left(xf_\xi + xf_\eta + yf_\zeta\right)\mathrm{d}x + 2\left(yf_\xi - yf_\eta + xf_\zeta\right)\mathrm{d}y$$

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$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

习题 9.3

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证明. 两边微分，得到

$$2(\mathrm{d}x + 2\,\mathrm{d}y - 3\,\mathrm{d}z)\cos(x+2y-3z) = \mathrm{d}x + 2\,\mathrm{d}y - 3\,\mathrm{d}z \Longrightarrow \mathrm{d}z = \frac{1}{3}\,\mathrm{d}x + \frac{2}{3}\,\mathrm{d}y$$

于是

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{1}{3} + \frac{2}{3} = 1$$

□

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证明. 两边微分，得到

$$c\varphi_1\,\mathrm{d}x - a\varphi_1\,\mathrm{d}z + c\varphi_2\,\mathrm{d}y - b\varphi_2\,\mathrm{d}z = 0 \Longrightarrow \mathrm{d}z = \frac{c\varphi_1}{a\varphi_1 + b\varphi_2}\,\mathrm{d}x + \frac{c\varphi_2}{a\varphi_1 + b\varphi_2}\,\mathrm{d}y$$

因此

$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial x} = \frac{ac\varphi_1}{a\varphi_1 + b\varphi_2} + \frac{bc\varphi_2}{a\varphi_1 + b\varphi_2}\,\mathrm{d}y = c$$

□

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由题

$$x^2-xy+y^2=1\Longrightarrow 2x\,\mathrm{d}x-y\,\mathrm{d}x-x\,\mathrm{d}y+2y\,\mathrm{d}y=0\Longrightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x-y}{x-2y}$$

且

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{(x-2y)(2-\frac{\mathrm{d}y}{\mathrm{d}x})-(2x-y)(1-2\frac{\mathrm{d}y}{\mathrm{d}x})}{(x-2y)^2}=-\frac{3y}{(x-2y)^2}+\frac{6x^2-3xy}{(x-2y)^3}$$

因此

$$\frac{\mathrm{d}z}{\mathrm{d}x}=2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x^2-2y^2}{x-2y}$$

进一步

$$\frac{\mathrm{d}^2z}{\mathrm{d}x^2}=2+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{4x-2y}{x-2y}+\frac{6x}{(x-2y)^3}$$

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联立消去 y，解得

$$x^2+(x+z)^2+z^2=1\Longrightarrow 2x\,\mathrm{d}x+2(x+z)\,\mathrm{d}x+2(x+z)\,\mathrm{d}z+2z\,\mathrm{d}z=0\Longrightarrow\frac{\mathrm{d}x}{\mathrm{d}z}=-\frac{x+2z}{2x+z}$$

同理

$$\frac{\mathrm{d}y}{\mathrm{d}z}=-\frac{y+2z}{2y+z}$$

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(1)

由题

$$\left.\begin{aligned}u^2+v^2+x^2+y^2=1\\u+v+x+y=0\end{aligned}\right\}\Longrightarrow\begin{cases}u\,\mathrm{d}u+v\,\mathrm{d}v+x\,\mathrm{d}x+y\,\mathrm{d}y=0\\\mathrm{d}u+\mathrm{d}v+\mathrm{d}x+\mathrm{d}y=0\end{cases}$$

联立解得

$$\begin{aligned}\mathrm{d}u&=\frac{x-v}{u-v}\,\mathrm{d}x+\frac{y-v}{u-v}\,\mathrm{d}y\\\mathrm{d}v&=\frac{x-u}{v-u}\,\mathrm{d}x+\frac{y-u}{v-u}\,\mathrm{d}y\end{aligned}$$

于是

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x}&\frac{\partial u}{\partial y}\\\frac{\partial v}{\partial x}&\frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}\frac{x-v}{u-v}&\frac{y-v}{u-v}\\\frac{x-u}{v-u}&\frac{y-u}{v-u}\end{vmatrix}=\frac{x-y}{u-v}$$

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两边微分，得到

$$\left.\begin{aligned} z &= xf(x+y) \\ 0 &= F(x,y,z) \end{aligned}\right\} \Longrightarrow \begin{cases} \mathrm{d}z = f(x+y)\,\mathrm{d}x + xf'(x+y)(\mathrm{d}x+\mathrm{d}y) \\ 0 = F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z \end{cases}$$

联立消去 $\mathrm{d}y$，解得

$$-F_z\left(\mathrm{d}z - (f(x+y)+xf'(x+y))\,\mathrm{d}x\right) = xf'(x+y)(F_x\,\mathrm{d}x + F_z\,\mathrm{d}z)$$

即

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{F_z f(x+y) + xF_z f'(x+y) - xF_x f'(x+y)}{F_z(1+xf'(x+y))}$$

习题 9.4

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由题

$$\mathbf{r}(t) = (a\cos t, a\sin t, bt) \Longrightarrow \mathbf{r}'(t) = (-a\sin t, a\cos t, b)$$

于是夹角

$$\theta = \arccos\frac{\mathbf{r}'(t)\cdot\mathbf{e}_3}{|\mathbf{r}'(t)|} = \arccos\frac{b}{a^2+b^2}$$

是定值。

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设 $\mathbf{r}(t_1) = \mathbf{r}(t_2)$，则 $t_1 = \pm t_2$。而由题 $t_1, t_2 > 0$，因此 $t_1 = t_2$，这是一条简单曲线。

进一步，不难得到

$$\mathbf{r}'(t) = \left(\frac{1}{(t+1)^2}, -\frac{1}{t^2}, 2t\right) \qquad \mathbf{r}''(t) = \left(-\frac{2}{(t+1)^3}, \frac{2}{t^3}, 2\right)$$
$$\mathbf{r}^{(n)}(t) = \left(\frac{(-1)^{n+1}n!}{(t+1)^{n+1}}, -\frac{(-1)^{n+1}n!}{t^{n+1}}, 0\right),\ n \geq 3$$

均连续，因此曲线光滑。

代入 $t=1$，知切线为

$$4\left(x-\frac{1}{2}\right) = -(y-2) = \frac{z-1}{2}$$

设法平面为 $x - 4y + 8z + d = 0$，代入 $\mathbf{r}(1)$，得到 $d = -\frac{1}{2}$，因此法平面为

$$2x - 8y + 16z - 1 = 0$$

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(1)

曲面可参数化为

$$\mathbf{r}(x,y)=\left(x,y,\sqrt{x^2+y^2}-xy\right)$$

于是

$$\mathbf{r}_x=\left(1,0,\frac{x}{\sqrt{x^2+y^2}}-y\right)\Longrightarrow\mathbf{r}_x(3,4)=\left(1,0,-\frac{17}{5}\right)$$
$$\mathbf{r}_y=\left(0,1,\frac{y}{\sqrt{x^2+y^2}}-x\right)\Longrightarrow\mathbf{r}_y(3,4)=\left(0,1,-\frac{11}{5}\right)$$

因此 $(3,4,-7)$ 处法向量为

$$\mathbf{n}=\left(1,0,-\frac{17}{5}\right)\times\left(0,1,-\frac{11}{5}\right)=\left(\frac{17}{5},\frac{11}{5},1\right)$$

法线为

$$\frac{x-3}{17}=\frac{y-4}{11}=\frac{z+7}{5}$$

进一步，设切平面为 $17x+11y+5z+d=0$，代入 $(3,4,-7)$ 解得 $d=-60$。于是切平面方程为

$$17x+11y+5z-60=0$$

(4)

对于隐式曲面

$$F(x,y,z)=\sqrt{x^2+y^2+z^2}-(x+y+z)+4=0$$

于是

$$F_x=\frac{x}{\sqrt{x^2+y^2+z^2}}-1\Longrightarrow F_x(2,3,6)=-\frac{5}{7}$$
$$F_y=\frac{y}{\sqrt{x^2+y^2+z^2}}-1\Longrightarrow F_y(2,3,6)=-\frac{4}{7}$$
$$F_z=\frac{z}{\sqrt{x^2+y^2+z^2}}-1\Longrightarrow F_z(2,3,6)=-\frac{1}{7}$$

因此 $(2,1,0)$ 处法向量为

$$\mathbf{n}=(5,4,1)$$

法线为

$$\frac{x-2}{5}=\frac{y-3}{4}=z-6$$

进一步，设切平面为 $5x+4y+z+d=0$，代入 $(2,3,6)$ 解得 $d=-28$。于是切平面方程为

$$5x+4y+z-28=0$$

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由题，椭球面可写成隐式曲面

$$F(x,y,z)=x^2+2y^2+3z^2-21=0$$

由

$$F_x=2x \qquad F_y=4y \qquad F_z=6z$$

知 (x_0,y_0,z_0) 处的切平面方程为

$$x_0(x-x_0)+2y_0(y-y_0)+3z_0(z-z_0)=0$$

展开得

$$x_0x+2y_0y+3z_0z=x_0^2+2y_0^2+3z_0^2=21$$

任取直线 L 上两点 $(6,3,\frac{1}{2})$ 和 $(0,0,\frac{7}{2})$，代入切平面方程，得到

$$z_0=\frac{7}{2}\Longrightarrow 6x_0+6y_0=21-\frac{3}{2}z_0=\frac{63}{4}$$

再结合

$$x_0^2+2y_0^2+z_0^2=21$$

解得

$$(x_0,y_0,z_0)=(3,0,2) \text{ 或 } (x_0,y_0,z_0)=(1,2,2)$$

进而切平面方程为

$$x+2z=7 \text{ 或 } x+4y+6z=21$$

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(1)

对于隐式曲线

$$F(x,y)=x^3y+xy^3+x^2+y^2-3=0$$

有

$$F_x=3x^2y+y^3+2xy^2\Longrightarrow F_x(1,1)=6$$
$$F_y=x^3+3xy^2+2x^2y\Longrightarrow F_y(1,1)=6$$

故 $(1,1)$ 处法向量为 $\mathbf{n}=(1,1)$，进而切向量为 $\mathbf{t}=(1,-1)$。

进一步，切线和法线依次为

$$y=-x+2 \qquad y=x$$

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(2)

考虑隐式曲面

$$F(x,y,z)=2x^2+3y^2+z^2-47=0 \qquad G(x,y,z)=x^2+2y^2-z=0$$

不难得到它们在 $(-2,1,6)$ 的法向量分别为

$$\mathbf{n}_1=(-4,3,6) \qquad \mathbf{n}_2=(-4,4,-1)$$

于是曲线的切向量为

$$\mathbf{t}=\mathbf{n}_1\times\mathbf{n}_2=(-27,-28,-4)$$

因此 $(1,3,4)$ 处切线为

$$\frac{x+2}{27}=\frac{y-1}{28}=\frac{z-6}{4}$$

进一步，设切平面为 $27x+28y+4z+d=0$，代入 $(-2,1,6)$ 解得 $d=2$。于是切平面方程为

$$27x+28y+4z+2=0$$

问题反馈

- 方向导数最大值是梯度模长，而不是某个绝对值最大的分量的绝对值；
- 结果要进行一定化简，方便拆的括号要拆，非必要不分式套分式，直线和平面方程要约分；
- 求 z 对 x 的导数，能不带 y 就不带 y；
- 椭球的法向量同向的切平面有两个。