第六周作业答案

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习题 9.5

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两边微分，得到

$$3z^2\,\mathrm{d}z - 2z\,\mathrm{d}x - 2x\,\mathrm{d}z + \mathrm{d}y = 0 \Longrightarrow \mathrm{d}z = \frac{2z}{3z^2-2x}\,\mathrm{d}x - \frac{1}{3z^2-2x}\,\mathrm{d}y$$

即

$$\frac{\partial z}{\partial x} = \frac{2z}{3z^2-2x} \qquad \frac{\partial z}{\partial y} = -\frac{1}{3z^2-2x}$$

进一步

$$\begin{aligned}
\mathrm{d}\frac{\partial z}{\partial x} &= \frac{2(3z^2-2x)\,\mathrm{d}z - 2z(6z\,\mathrm{d}z - 2\,\mathrm{d}x)}{(3z^2-2x)^2} \\
&= \frac{2}{3z^2-2x}\,\mathrm{d}z - \frac{12z^2}{(3z^2-2x)^2}\,\mathrm{d}z + \frac{4z}{(3z^2-2x)^2}\,\mathrm{d}x \\
&= -\frac{6z^2+4x}{(3z^2-2x)^2}\left(\frac{2z}{3z^2-2x}\,\mathrm{d}x - \frac{1}{3z^2-2x}\,\mathrm{d}y\right) + \frac{4z}{(3z^2-2x)^2}\,\mathrm{d}x \\
&= -\frac{16xz}{(3z^2-2x)^3}\,\mathrm{d}x + \frac{6z^2+4x}{(3z^2-2x)^3}\,\mathrm{d}y
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}\frac{\partial z}{\partial y} &= \frac{6z\,\mathrm{d}z - 2\,\mathrm{d}x}{(3z^2-2x)^2} \\
&= -2(3z^2-2x)^2\,\mathrm{d}x + \frac{6z}{(3z^2-2x)^2}\left(\frac{2z}{3z^2-2x}\,\mathrm{d}x - \frac{1}{3z^2-2x}\,\mathrm{d}y\right) \\
&= \frac{6z^2+4x}{(3z^2-2x)^3}\,\mathrm{d}x - \frac{6z}{(3z^2-2x)^3}\,\mathrm{d}y
\end{aligned}$$

即

$$\frac{\partial^2 z}{\partial x^2} = -\frac{16xz}{(3z^2-2x)^3} \qquad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = \frac{6z^2+4x}{(3z^2-2x)^3} \qquad \frac{\partial^2 z}{\partial y^2} = -\frac{6z}{(3z^2-2x)^3}$$

代入 $(x,y,z)=(1,1,1)$，得到展开式

$$z(x,y) = 1 + 2(x-1) - (y-1) - 8(x-1)^2 + 10(x-1)(y-1) - 3(y-1)^2 + o(\rho^2)$$

其中 $\rho = \sqrt{(x-1)^2+(y-1)^2}$。

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(3)

考虑函数

$$F(x,y,z) = \sin x \sin y \sin z + \lambda\left(x+y+z-\frac{\pi}{2}\right)$$

则

$$\begin{cases} F_x(x,y,z) = \cos x \sin y \sin z + \lambda = 0 \\ F_y(x,y,z) = \sin x \cos y \sin z + \lambda = 0 \\ F_z(x,y,z) = \sin x \sin y \cos z + \lambda = 0 \\ x+y+z = \dfrac{\pi}{2} \end{cases}$$

解得

$$x = y = z = \frac{\pi}{6}$$

进一步，$(x,y,z) = (\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ 时

$$\begin{pmatrix} u_{xx} & u_{xy} & u_{xz} \\ u_{xy} & u_{yy} & u_{yz} \\ u_{zx} & u_{zy} & u_{zz} \end{pmatrix} = \begin{pmatrix} -\sin x \sin y \sin z & \cos x \cos y \sin z & \cos x \sin y \cos z \\ \cos x \cos y \sin z & -\sin x \sin y \sin z & \sin x \cos y \cos z \\ \cos x \sin y \cos z & \sin x \cos y \cos z & -\sin x \sin y \sin z \end{pmatrix} = \frac{1}{8}\begin{pmatrix} -1 & 3 & 3 \\ 3 & -1 & 3 \\ 3 & 3 & -1 \end{pmatrix} < 0$$

于是 $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ 显然是极大值点，极大值为 $\frac{1}{8}$。

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设张成该平行六面体的三个向量为 $\mathbf{a}, \mathbf{b}, \mathbf{c}$，先固定它们的模长。此时体积为 $|\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}|$。

于是在 $\mathbf{a}, \mathbf{b}$ 方向固定时 $\mathbf{c}$ 与 $\mathbf{a} \times \mathbf{b}$ 共线时体积最大。此时 $\mathbf{c} \cdot \mathbf{a} = \mathbf{c} \cdot \mathbf{b} = 0$，体积为 $|\mathbf{a} \times \mathbf{b}||\mathbf{c}|$。

进一步，由 $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta \le |\mathbf{a}||\mathbf{b}|$，知体积最大时 $\mathbf{a}, \mathbf{b}, \mathbf{c}$ 两两垂直。

下面，设共顶点的三条棱长度分别为 x, y, z，则 $x+y+z = 3a$，体积为 $V = xyz$。

考虑函数

$$f(x,y,z) = xyz + \lambda(x+y+z-3a) \tag{1}$$

则

$$\begin{cases} f_x(x,y,z) = yz + \lambda = 0 \\ f_y(x,y,z) = xz + \lambda = 0 \\ f_z(x,y,z) = xy + \lambda = 0 \\ x+y+z = 3a \end{cases}$$

解得 $x = y = z$，即 $x = y = z = a$，此时体积最大，为 a^3。

习题 10.1

1

(1)

$$\int_0^1 \mathrm{d}y \int_0^{\sqrt{1-x^2}} f(x,y)\,\mathrm{d}y = \int_{-1}^1 \mathrm{d}y \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,\mathrm{d}x$$

(3)

$$\int_0^a \mathrm{d}y \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\,\mathrm{d}x = \int_0^{2a} \mathrm{d}x \int_0^{\sqrt{2ax-x^2}} f(x,y)\,\mathrm{d}y$$

(5)

$$\int_0^1 \mathrm{d}x \int_0^x f(x,y)\,\mathrm{d}y + \int_1^2 \mathrm{d}y \int_0^{2-x} f(x,y)\,\mathrm{d}x = \int_0^1 \mathrm{d}x \int_y^{2-y} f(x,y)\,\mathrm{d}y$$

2

(1)

令 $t=y^2$，则

$$\begin{aligned}
\iint_D \frac{y}{(1+x^2+y^2)^{\frac{3}{2}}}\,\mathrm{d}x\,\mathrm{d}y &= \int_0^1 \mathrm{d}x \int_0^1 \frac{y}{(1+x^2+y^2)^{\frac{3}{2}}}\,\mathrm{d}y \\
&= \frac{1}{2}\int_0^1 \mathrm{d}x \int_0^1 \frac{1}{(1+x^2+t)^{\frac{3}{2}}}\,\mathrm{d}t \\
&= \int_0^1 \left(\frac{1}{(1+x^2)^{\frac{1}{2}}} - \frac{1}{(2+x^2)^{\frac{1}{2}}} \right) \mathrm{d}x \\
&= \ln(1+\sqrt{2}) - \ln\frac{1+\sqrt{3}}{\sqrt{2}}
\end{aligned}$$

(2)

$$\begin{aligned}
\iint_D \sin(x+y)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^\pi \mathrm{d}x \int_0^\pi \sin(x+y)\,\mathrm{d}y \\
&= \int_0^\pi (\cos x - \cos(x+\pi))\,\mathrm{d}x \\
&= 2\int_0^\pi \cos x\,\mathrm{d}x \\
&= 0
\end{aligned}$$

(5)

$$
\begin{aligned}
\iint_D (x+y-1)\,\mathrm{d}x\,\mathrm{d}y &= \int_a^{3a} \mathrm{d}y \int_{y-a}^{y} (x+y-1)\,\mathrm{d}x \\
&= \int_a^{3a} \left(ay + \frac{1}{2}y^2 - \frac{1}{2}(y-a)^2 - a \right) \mathrm{d}y \\
&= \int_a^{3a} \left(2ay - \frac{1}{2}a^2 - a \right) \mathrm{d}y \\
&= 7a^3 - 2a^2
\end{aligned}
$$

(6)

$$
\iint_D \frac{\sin y}{y}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \mathrm{d}y \int_{y^2}^{y} \frac{\sin y}{y}\,\mathrm{d}x = \int_0^1 (\sin y - y\sin y)\,\mathrm{d}y = 1 - \sin 1
$$

(8)

记

$$
D_1 = \left\{ (x,y) \in D | x+y \le \frac{\pi}{2} \right\} \qquad D_2 = \left\{ (x,y) \in D | x+y > \frac{\pi}{2} \right\}
$$

则

$$
\begin{aligned}
\iint_D |\cos(x+y)|\,\mathrm{d}x\,\mathrm{d}y &= \iint_{D_1} \cos(x+y)\,\mathrm{d}x\,\mathrm{d}y - \iint_{D_2} \cos(x+y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^{\frac{\pi}{4}} \mathrm{d}y \int_x^{\frac{\pi}{2}-x} \cos(x+y)\,\mathrm{d}x - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \mathrm{d}x \int_{\frac{\pi}{2}-x}^{x} \cos(x+y)\,\mathrm{d}y \\
&= \int_0^{\frac{\pi}{4}} (1 - \sin 2y)\,\mathrm{d}y - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin 2x - 1)\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} (1 - \sin 2y)\,\mathrm{d}y \\
&= \frac{\pi}{2} - 1
\end{aligned}
$$

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证明. 只要 f 在 $[a,b] \times [c,d]$ 可积，就有

$$
\iint_{[a,b]\times[c,d]} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \mathrm{d}x \int_c^d \varphi(x)\psi(y)\,\mathrm{d}y = \int_a^b \varphi(x)\,\mathrm{d}x \int_c^d \psi(y)\,\mathrm{d}y = \int_a^b \varphi(x)\,\mathrm{d}x \int_c^d \psi(x)\,\mathrm{d}x
$$

下面只要证可积性。

事实上，对于分割

$$
\pi_1 : a = x_0 < x_1 < \cdots < x_m = b
$$

$$
\pi_2 : a = y_0 < y_1 < \cdots < y_n = b
$$

有

$$
\begin{aligned}
&\lim_{\substack{\|\pi_1\|\to 0\\ \|\pi_2\|\to 0}} \sum_{i=1}^{m}\sum_{j=1}^{n} f(\xi_i,\eta_j)(x_i-x_{i-1})(y_j-y_{j-1})\\
=&\lim_{\substack{\|\pi_1\|\to 0\\ \|\pi_2\|\to 0}} \sum_{i=1}^{m}\sum_{j=1}^{n} \varphi(\xi_i)\psi(\eta_j)(x_i-x_{i-1})(y_j-y_{j-1})\\
=&\lim_{\substack{\|\pi_1\|\to 0\\ \|\pi_2\|\to 0}} \sum_{i=1}^{m} \varphi(\xi_i)(x_i-x_{i-1})\sum_{j=1}^{n}\psi(\eta_j)(y_j-y_{j-1})\\
=&\left(\lim_{\|\pi_1\|\to 0}\sum_{i=1}^{m}\varphi(\xi_i)(x_i-x_{i-1})\right)\left(\lim_{\|\pi_2\|\to 0}\sum_{j=1}^{n}\psi(\eta_j)(y_j-y_{j-1})\right)\\
=&\int_a^b \varphi(x)\,\mathrm{d}x\int_c^d \psi(y)\,\mathrm{d}y
\end{aligned}
$$

存在，从而 f 在 $[a,b]\times[c,d]$ 可积。 □

问题反馈

- 刚开始学累次积分换序时，可以多画画图，想象被积区域的形状，把上下限写对；
- 上课没讲过的函数，如 $\sinh^{-1}$，尽量不要用；
- 目前所学知识，可积性的等价判定只有分割求和取极限。