第三周作业答案

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习题 9.2

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(2)

$$\frac{\partial z}{\partial x} = \frac{3^{-\frac{y}{x}} y \ln 3}{x^2} \qquad \frac{\partial z}{\partial y} = -\frac{3^{-\frac{y}{x}} \ln 3}{x}$$

(5)

$$\frac{\partial u}{\partial x} = \frac{(x-y)^2}{2x^2+2y^2} \frac{-2y}{(x-y)^2} = -\frac{y}{x^2+y^2} \qquad \frac{\partial u}{\partial y} = \frac{(x-y)^2}{2x^2+2y^2} \frac{2x}{(x-y)^2} = \frac{x}{x^2+y^2}$$

(7)

$$\frac{\partial u}{\partial x} = y^z x^{y^z-1} \qquad \frac{\partial u}{\partial y} = zy^{z-1} x^{y^z} \ln x \qquad \frac{\partial u}{\partial z} = y^z x^{y^z} \ln x \ln y$$

(8)

$$\frac{\partial u}{\partial x} = e^{-z} + \frac{1}{x+\ln y} \qquad \frac{\partial u}{\partial y} = \frac{1}{y(x+\ln y)} \qquad \frac{\partial u}{\partial z} = -xe^{-z} + 1$$

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$$\frac{\partial f}{\partial x} = \frac{2\sin x^2 y}{x} \qquad \frac{\partial f}{\partial y} = \frac{\sin x^2 y}{y}$$

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x 方向的偏导数

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} y \sin \frac{1}{x^2+y^2}\bigg|_{y=0} = 0$$

y 方向的偏导数

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} y \sin \frac{1}{x^2+y^2}\bigg|_{x=0} = \lim_{y\to 0} \sin \frac{1}{y^2}$$

即 y 方向的偏导数不存在。

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由题

$$\frac{\partial z}{\partial x}\bigg|_{(x,y)=(2,4)} = \left(\frac{\partial}{\partial x}\ \frac{x^2+y^2}{4}\bigg|_{y=4}\right)\bigg|_{x=2} = \left(\frac{\partial}{\partial x}\frac{x^2+16}{4}\right)\bigg|_{x=2} = 1$$

因此夹角为 $\frac{\pi}{4}$。

8

不难得到

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -\frac{1}{2t^{\frac{3}{2}}} e^{-\frac{x^2}{4t}} + \frac{x^2}{4t^{\frac{5}{2}}} e^{-\frac{x^2}{4t}} \\
\frac{\partial u}{\partial x} &= -\frac{x}{2t^{\frac{3}{2}}} e^{-\frac{x^2}{4t}} \\
\frac{\partial^2 u}{\partial x^2} &= -\frac{1}{2t^{\frac{3}{2}}} e^{-\frac{x^2}{4t}} + \frac{x^2}{4t^{\frac{5}{2}}} e^{-\frac{x^2}{4t}}
\end{aligned}$$

代入验证即可。

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(1)

证明.

$$\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) = \frac{y^2+z^2}{r^{\frac{3}{2}}} \Longrightarrow \Delta r = \frac{2}{r}$$

□

(2)

证明.

$$\frac{\partial^2 \ln r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right) = \frac{y^2+z^2-x^2}{r^4} \Longrightarrow \Delta \ln r = \frac{1}{r^2}$$

□

(3)

证明.
$$\frac{\partial^2}{\partial x^2}\frac{1}{r} = -\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) = \frac{y^2+z^2-2x^2}{r^{\frac{5}{2}}} \Longrightarrow \Delta\frac{1}{r} = 0$$

□

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(4)

$$\left.\begin{aligned}\frac{\partial z}{\partial x} = -\frac{y}{x^2+y^2}\\ \frac{\partial z}{\partial \partial} = \frac{x}{x^2+y^2}\end{aligned}\right\} \Longrightarrow \mathrm{d}z = -\frac{y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y$$

(6)

$$\left.\begin{aligned}\frac{\partial z}{\partial x} = 4x^3-8xy^2\\ \frac{\partial z}{\partial y} = 4y^3-8x^2y\end{aligned}\right\} \Longrightarrow \mathrm{d}z = y\cos xy\,\mathrm{d}x + x\cos xy\,\mathrm{d}y \Longrightarrow \begin{cases}\mathrm{d}z(0,0)=0\\ \mathrm{d}z(1,1) = -4\,\mathrm{d}x - 4\,\mathrm{d}y\end{cases}$$

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证明. 假设 $f(x,y)=\sqrt{|xy|}$ 在原点可微，则

$$f(h,k)-f(0,0) = \sqrt{|hk|} = ah+bk+o\left(\sqrt{h^2+k^2}\right)$$

由对称性，只能有 $a=b$。

再取 $k=-h$，得到

$$|h| = (a-b)h + o(|h|) = o(|h|)$$

矛盾！

□

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证明. 由

$$0 \le \lim_{\substack{x\to 0\\ y\to 0}}\left|\frac{x^2y}{x^2+y^2}\right| \le \lim_{\substack{x\to 0\\ y\to 0}}\frac{|x|}{2} = 0$$

知 $f(x,y)$ 在 $(0,0)$ 连续。

进一步，$(0,0)$ 处偏导数

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0}\left.\frac{xy}{x^2+y^2}\right|_{y=0} = 0 \qquad \frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0}\left.\frac{y^2}{x^2+y^2}\right|_{y=0} = 0$$

在 $(0,0)$ 均存在。

另一方面，假设可微，则极限

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{f(x,y)}{\sqrt{x^2+y^2}} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2y}{(x^2+y^2)^{\frac{3}{2}}}$$

存在。但是

$$\begin{aligned}
\lim_{\substack{x\to 0\\ y=0}} \frac{x^2y}{(x^2+y^2)^{\frac{3}{2}}} &= 0 \\
\lim_{\substack{x\to 0\\ y=x}} \frac{x^2y}{(x^2+y^2)^{\frac{3}{2}}} &= \frac{1}{2\sqrt{2}}
\end{aligned}$$

矛盾！ □

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(2)

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = f_x \cos t - f_y \sin t + e^t f_z$$

(3)

$$\begin{aligned}
\frac{\partial u}{\partial x} &= 2xf_1 + ye^{xy} f_2 \\
\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial x}(-2yf_1 + xe^{xy} f_2) \\
&= -4xyf_{11} + 2(x^2-y^2)e^{xy} f_{12} + (x+y)e^{xy} f_2 + xye^{2xy} f_{22}
\end{aligned}$$

(4)

$$\begin{aligned}
\frac{\partial u}{\partial x} &= f_1 + 2xf_2 \\
\frac{\partial^2 u}{\partial x^2} &= f_{11} + 4xf_{12} + 2f_2 + 4x^2 f_{22} \\
\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial x}(f_1 + 2yf_2) = f_{11} + 2(x+y)f_{12} + 4xyf_{22}
\end{aligned}$$

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$$
\begin{aligned}
\frac{\partial u}{\partial x} &= y\varphi_1 f'(t) + \varphi_2 f'(t) \\
\frac{\partial u}{\partial y} &= x\varphi_1 f'(t) + \varphi_2 f'(t) \\
\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(x\varphi_1 f'(t) + \varphi_2 f'(t)\right) \\
&= \varphi_1 f'(t) + xy\varphi_{11} f'(t) + x\varphi_{12} f'(t) + xy\varphi_1^2 f''(t) + x\varphi_1\varphi_2 f''(t) \\
&+ y\varphi_{21} f'(t) + \varphi_{22} f'(t) + y\varphi_1\varphi_2 f''(t) + \varphi_2^2 f''(t) \\
&= f'(t)\left(\varphi_1 + xy\varphi_{11} + x\varphi_{12} + y\varphi_{21} + \varphi_{22}\right) + f''(t)\left(xy\varphi_1^2 + (x+y)\varphi_1\varphi_2 + \varphi_2^2\right) \\
&= f'(t)\left(\varphi_1 + xy\varphi_{11} + (x+y)\varphi_{12} + \varphi_{22}\right) + f''(t)\left(xy\varphi_1^2 + (x+y)\varphi_1\varphi_2 + \varphi_2^2\right)
\end{aligned}
$$

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证明.

$$
x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = xyf'(xy) - yxf'(xy) = 0
$$

□

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证明.

$$
x\frac{\partial z}{\partial x} - y^2\frac{\partial z}{\partial y} = f'\left(\ln x + \frac{1}{y}\right) - f'\left(\ln x + \frac{1}{y}\right) = 0
$$

□

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证明.

$$
\left.\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= a^2\varphi''(x-at) + a^2\psi''(x+at) \\
\frac{\partial^2 u}{\partial x^2} &= \varphi''(x-at) + \psi''(x+at)
\end{aligned}\right\} \Longrightarrow \frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}
$$

□

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证明. 由

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \\
\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = -2\frac{\partial z}{\partial u} + a\frac{\partial z}{\partial v}
\end{aligned}
$$

知

$$
\begin{aligned}
0 &= 6\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2} \\
&= 6\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial x}\left(-2\frac{\partial z}{\partial u} + a\frac{\partial z}{\partial v}\right) - \frac{\partial}{\partial y}\left(-2\frac{\partial z}{\partial u} + a\frac{\partial z}{\partial v}\right) \\
&= 6\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-2\frac{\partial z}{\partial u} + a\frac{\partial z}{\partial v}\right) - \left(-2\frac{\partial}{\partial u} + a\frac{\partial}{\partial v}\right)\left(-2\frac{\partial z}{\partial u} + a\frac{\partial z}{\partial v}\right) \\
&= (10 + 5a)\frac{\partial^2 z}{\partial u \partial v} - (6 + a - a^2)\frac{\partial^2 z}{\partial v^2}
\end{aligned}
$$

于是 $a = 3$。 □

习题 9.3

1

(1)

证明. 对于

$$
F(x, y) = x^2 + xy + y^2 - 7 \in C^1(\mathbb{R}^2)
$$

有

$$
F(2, 1) = 0 \qquad F_y(2, 1) = 4 \neq 0
$$

由隐函数存在定理，知该方程在 $(2, 1)$ 附近有唯一解 $y(x)$。

两边微分，得到

$$
2x\,\mathrm{d}x + y\,\mathrm{d}x + x\,\mathrm{d}y + 2y\,\mathrm{d}y = 0 \Longrightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x + y}{x + 2y} \Longrightarrow y'(2) = -\frac{5}{4}
$$

进一步

$$
\begin{aligned}
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= -\frac{(2\,\mathrm{d}x + \mathrm{d}y)(x + 2y) - (2x + y)(\mathrm{d}x + 2\,\mathrm{d}y)}{(x + 2y)^2\,\mathrm{d}x} \\
&= \frac{(2x + y) - 2(x + 2y)}{(x + 2y)^2} + \frac{2(2x + y) - (x + 2y)}{(x + 2y)^2}\frac{\mathrm{d}y}{\mathrm{d}x} \\
&= -\frac{6(x^2 + xy + y^2)}{(x + 2y)^3} \\
&\Longrightarrow y''(2) = -\frac{21}{32}
\end{aligned}
$$

□

2

(2)

两边微分，得到

$$\frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{x^2+y^2} = \frac{x^2}{x^2+y^2}\left(-\frac{y}{x^2}\,\mathrm{d}x + \frac{1}{x}\,\mathrm{d}y\right)$$

整理得

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y}{x-y}$$

进一步

$$\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{1}{(x-y)^2}\left((\mathrm{d}x+\mathrm{d}y)(x-y)-(x+y)(\mathrm{d}x-\mathrm{d}y)\right) = \frac{-2y}{(x-y)^2}\,\mathrm{d}x + \frac{2x}{(x-y)^2}\,\mathrm{d}y$$

代入 $\mathrm{d}y = \frac{x+y}{x-y}$ 得

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{2y}{(x-y)^2} + \frac{2y(x+y)}{(x-y)^3} = \frac{4y^2}{(x-y)^3}$$

(5)

两边微分，得到

$$\frac{1}{z}\,\mathrm{d}x - \frac{x}{z^2}\,\mathrm{d}z = \frac{1}{z}\,\mathrm{d}z - \frac{1}{y}\,\mathrm{d}y$$

整理得

$$\frac{\partial z}{\partial x} = \frac{z}{x+z} \qquad \frac{\partial z}{\partial y} = \frac{z^2}{y(x+z)}$$

4

(1)

两边微分，得到

$$-\sin x\cos x\,\mathrm{d}x - \sin y\cos y\,\mathrm{d}y - \sin z\cos z\,\mathrm{d}z = 0 \Longrightarrow \mathrm{d}z = -\frac{\sin 2x}{\sin 2z}\,\mathrm{d}x - \frac{\sin 2y}{\sin 2z}\,\mathrm{d}y$$

问题反馈

- 对于 $u = f$ 形的函数求各阶偏导数，最终的结果不应带 u，而应该带 f；
- 以习题 9.2 的 20(4) 为例，f 的偏导那项，**一定不能**写成 $\frac{\partial f}{\partial(x+y+z)}$。一种写法是单独设出新变量 $\xi = x+y+z$，然后分母改为 ξ；另一种写法是直接 f_1，意为对第一分量求导。
- 求“一点处”的微分/偏导数时要把那点代入算出来；
- 能写弧度制就不要写角度制。